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Higher spins and stringy $AdS_5 \times S^{5*}$

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Abstract

In this lecture I review recent work done in collaboration with N. Beisert, J. F. Morales and H. Samtleben [1–3][1]. After a notational flash on the AdS/CFT correspondence, I will discuss higher spin (HS) symmetry enhancement at small radius and how this is holographically captured by free N=4 SYM theory. I will then derive the spectrum of perturbative superstring excitations on AdS in this particular limit and successfully compare it with the spectrum of single-trace operators in free $\mathcal{N} = 4$ SYM at large N, obtained by means of Polya(kov)'s counting. Decomposing the spectrum into HS multiplets allows one to precisely identify the 'massless' HS doubleton and the lower spin Goldstone multiplets which participate in the pantagruelic Higgs mechanism, termed *"La Grande Bouffe"*. After recalling some basic features of Vasiliev's formulation of HS gauge theories, I will eventually sketch how to describe mass generation in the AdS bulk à la Sückelberg and its holographic implications such as the emergence of anomalous dimensions in the boundary $\mathcal{N} = 4$ SYM theory.

1 Introduction and Summary

The plan of the lecture is as follows.

I will begin with a flash on the AdS/CFT correspondence with the purpose of establishing the notation and recalling how semiclassical string solitons with large spin (S)

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[1]For a concise summary see [4].

and/or R-charge (J) in the bulk can be associated to gauge-invariant composite operators on the boundary, for reviews see *e.g.* [5–8] and references therein.

I will then pass to consider Higher Spin (HS) Symmetry enhancement at vanishing coupling on the boundary and discuss how this regime should be captured by the extreme stringy regime of very 'small' radius. Exploiting HS enhancement and extrapolating the BMN formula, I will show how to derive the spectrum of perturbative superstring excitations on AdS in this particular limit and compare it with the spectrum of single-trace operators in free $\mathcal{N} = 4$ SYM at large N, obtained by means of Polya(kov)'s counting. Needless to say perfect agreement is found up to the level that is possible to reach by computer-aided human means.

Decomposing the spectrum into HS multiplets allows one to precisely identify the 'massless' HS doubleton, comprising the HS currents on the boundary dual to massless – in this limit – gauge fields in the bulk, with the 'first Regge trajectory'. Higher Regge trajectories correspond to 'massive' HS L-pletons (YT-pletons), associated to Yang-tableaux compatible with gauge invariance, comprise KK excitations of the doubleton, lower spin Goldstone modes as well as genuinely long superconformal multiplets.

Glimpses of *"La Grande Bouffe"*[2], the Pantagruelic Higgs mechanism whereby HS gauge fields eat lower spin Goldstone fields and become massive, are presented in the Stückelberg formulation that suggests how the mass shifts should be governed by (broken) HS symmetry. I will not dwelve too much into the discussion of anomalous dimensions that emerge in the boundary theory when interactions are turned on as a result of the resolution of operator mixing.

2 Notational flash on AdS/CFT

Maldacena's conjectures triggered an intense renewal of interest in (super)conformal field theories (SCFT) and lead to the discovery of previously unknown non-renormalization properties [9–14]. In particular the holographic correspondence between $\mathcal{N} = 4$ SYM theory in $d = 4$ with $SU(N)$ gauge group and Type IIB superstring theory on $AdS^5 \times S^5$ with N units of RR 5-form flux has been an inhexaustible source of insights in the duality between gauge fields and strings. $\mathcal{N} = 4$ SYM theory is an exactly superconformal field theory at the quantum level. The elementary field content of the theory consists of one gauge vector A_μ, **4** Weyl gaugini λ^A_α together with their $\mathbf{4}^*$ conjugate $\bar{\lambda}^{\dot{\alpha}}_A$ and **6** real scalars φ^i all in the adjoint representation of the gauge group. All interactions

[2]Several people asked me the origin of this terminology: it is the title of a movie directed by Marco Ferreri, interpreted, among other, by Marcello Mastroianni and Ugo Tognazzi and presented in 1973 at *Festival du Cinema* in Cannes where it received the International Critics Award.

up to two derivatives are fixed by the choice of the gauge group, *i.e.* of the structure constants f_{abc}, and the β-function vanishes both perturbatively and non-perturbatively[3]. The semiclassical relation between the couplings $g^2_{ym} = 4\pi g_s$ and the curvature radius $R^4 = \lambda(\alpha')^2$, where $\lambda = g^2_{ym}N$ is the 't Hooft coupling, suggests that the planar limit, dominated by amplitudes with the topology of the sphere, is achieved at large N with fixed small λ. However one can trust the low-energy supergravity approximation only at large λ where the curvature is small, a regime that is difficult to analyze from the SYM perspective except for few observables protected against quantum corrections by extended superconformal symmetry. The relavant supergroup, $(P)SU(2,2|4)$, includes the $SU(2,2) \approx SO(4,2)$ isometry of AdS_5 acting as conformal group on the boundary, and the $SU(4) \approx SO(6)$ isometry of S^5, playing the role of R-symmetry group in $\mathcal{N} = 4$ SYM. Each gauge-invariant local composite operator $\mathcal{O}_\Delta(x)$ of scaling dimension Δ on the (conformal) boundary $\rho \approx 0$ of AdS is expected to be holographically dual to a bulk field $\Phi_M(x,\rho)$, associated to a string excitation of AdS mass M. The near boundary behaviour $\Phi(x,\rho) \approx \rho^{4-\Delta} j(x) + ...$ is dictated by the boundary source $j_{4-\Delta}(x)$ that couples to $\mathcal{O}_\Delta(x)$. Linearized field equations determine the mass-to-dimension relation

$$M^2R^2 = \Delta(\Delta - 4) - \Delta_u(\Delta_u - 4) \tag{2.1}$$

where Δ_u is the lower bound on Δ imposed by unitarity of the $(P)SU(2,2|4)$ representation $\Phi_M \approx \mathcal{O}_\Delta$ belongs to.

A very interesting class of operators consists of those in 1/2 BPS (ultra) short multiplets that correspond to the $\mathcal{N} = 8$ gauged supergravity multiplet and its Kaluza-Klein (KK) recurrences. Their lowest *superprimary* components are scalar Chiral Primary Operators (CPO's)

$$\mathcal{Q}^{(i_1 \dots i_p)|} = Tr(\varphi^{i_1} \dots \varphi^{i_p}) \tag{2.2}$$

of dimension $\Delta = p$ belonging to the p-fold totally symmetric and traceless tensor representation of the $SO(6)$ R-symmetry with $SU(4)$ Dynkin labels $[0,p,0]$. CPO's are not only annihilated by the 16 superconformal charges S^A_α, $\bar{S}^{\dot{\alpha}}_A$ but also by half of the 16 Poincarè supercharges Q^A_α, $\bar{Q}^{\dot{\alpha}}_A$. Generically, 1/2 BPS multiplets contain $2^8p^2(p^2-1)/12$ components. $p = 0$ is the identity while $p = 1$ is the *singleton* representation of $PSU(2,2|4)$ corresponding to the elementary SYM fields (8 bosons and 8 fermions) that do not propagate in the bulk. The $\mathcal{N} = 4$ supercurrent multiplet ($p = 2$) contains 128 bosonic and as many fermionic components and includes the conserved traceless stress tensor $\mathcal{T}_{\mu\nu}$, the **15** conserved R-symmetry currents $\mathcal{J}^{[ij]}_\mu$ and the $\mathbf{4}^*$ conserved and γ-traceless supercurrents

[3]Due to instanton effects, physical observables may *a priori* depend on θ_{ym} since $\mathcal{N} = 4$ SYM has no internal R-symmetry anomaly. Holography relates the vacuum angle to the axion $\theta_{ym} = 2\pi\chi$ and YM instantons to type IIB D-instantons [15–17, 19, 18, 20–23].

$\Sigma^{\alpha}_{\mu A}$ as well as their **4** conjugate $\bar{\Sigma}^{A}_{\mu\dot{\alpha}}$. Correlation functions of CPO's

$$G(x_1, ...x_n) = \langle \mathcal{Q}_{p_1}(x_1)\mathcal{Q}_{p_2}(x_2)...\mathcal{Q}_{p_n}(x_n)\rangle \tag{2.3}$$

enjoy remarkable (partial) non-renormalization properties [9–14]. Two and three-point functions as well as extremal ($p_1 = \sum_{i\neq 1} p_i$) and next-to-extremal ($p_1+2 = \sum_{i\neq 1} p_i$) correlators do not receive any quantum correction. Near extremal correlators ($p_1+k = \sum_{i\neq 1} p_i$, with small k), *e.g.* four-point functions of the CPO's $\mathcal{Q}_2$ in the supercurrent multiplet, display some sort of partial non-renormalization both at weak coupling perturbatively and non, where field theory methods are reliable, and at strong coupling where supergravity is reliable.

Massive string excitations correspond to long multiplets with at least 2^{16} components [24, 25]. The prototype is the $\mathcal{N} = 4$ Konishi multiplet [26] that starts with the scalar singlet operator $\mathcal{K} = Tr(\varphi_i \varphi^i)$ of naive dimension $\Delta_0 = 2$ at vanishing coupling where a unitary bound of the semishort kind is saturated and the currents of spin up to 4 at higher level are conserved. When interactions are turned on, all the components of the multiplet acquire the same anomalous dimension $\gamma^{\mathcal{K}}$, that was computed at one loop long ago [27]. The result, $\gamma^{\mathcal{K}}_{1-loop} = +3\lambda$, has been confirmed and extended to two loops, $\gamma^{\mathcal{K}}_{2-loop} = -3\lambda^2$, by explicit computations [26, 28] and three loops, $\gamma^{\mathcal{K}}_{3-loop} = 21/4\lambda^3$ [29, 30], assuming integrability of the super spin chain, whose hamiltonian represents the dilatation operator [31,29,32,33]. Instanton corrections are absent to lowest order [26,23]. At strong 't Hooft coupling operators dual to string excitations with masses $M^2 \approx 1/\alpha'$ should acquire large anomalous dimensions $\Delta = \Delta_0 + \gamma \approx MR \approx \lambda^{1/4}$ and decouple from the operator algebra. Unfortunately, it has been difficult to test and exploit the correspondence beyond the supergravity approximation since, despite some progress [34–39]

an efficient quantization scheme for type IIB superstring on $AdS_5 \times S^5$ is still lacking. Berkovits's pure spinor formalism [40] allows one to write covariant emission vertices for 'massless' supergravity fields and their KK recurrences [41]. Studying the first massive level, that, as we will argue, corresponds to the Konishi multiplet and its KK recurrences, should shed new light on the stringy aspects of the holographic correspondence.

Alternatively one can consider particular sectors of the spectrum or peculiar regimes where computations are feasible in both descriptions. On the one hand one can study semiclassical string solitons with large spin or R-charge, whose dynamics can be studied perturbatively in terms of a reduced coupling $\lambda' = \lambda/L^2$ where L measures the 'length' of the operator / string [8]. On the other hand one can try to study the string spectrum and interactions in the very stringy regime of small radius R, dual to free $\mathcal{N} = 4$ SYM, where holography predicts Higher Spin (HS) symmetry enhancement [42–45]. Before doing that let us briefly recall some results concerning string solitons in AdS. In QCD, processes like Deep Inelastic Scattering can be studied by means of Operator Products Expansions

(OPE's) of local operators that are dominated by operators with arbitrary dimension Δ and spin s but fixed twist $\tau = \Delta - s$, in particular currents with $\tau = 2 + \gamma$ such as

$$J^v_{(\mu_1\mu_2...\mu_s)|} = Tr(F_{\nu(\mu_1} D_{\mu_2} ... D_{\mu_{s-1}} F_{\mu_s)|}{}^{\nu}) \quad , \quad J^f_{(\mu_1\mu_2...\mu_s)|} = \bar{\psi}_u \gamma_{(\mu_1} D_{\mu_2} ... D_{\mu_s)|} \psi^u \tag{2.4}$$

These mix (beyond one-loop) with one another and, except for the stress tensor and the conserved vector currents, acquire anomalous dimensions $\gamma(S)$. At large S the dominant contribution, including higher loops, goes as $\gamma(S) \approx \log(S)$. This remarkable gauge theory prediction [46, 47] has been confirmed by string computations [48] showing that the dispersion relation for long folded strings in AdS_5 with large S is of the form

$$MR = \Delta = S + a\sqrt{\lambda}\log(S) + ... \tag{2.5}$$

Unfortunately it is difficult at present to quantitatively study the perturbative contributions in λ that should reconstruct the $\sqrt{\lambda}$ at large λ. It is however reassuring to observe that small strings, even in AdS_5, display the standard relation $M^2 = S/\alpha'$.

In addition to the above HS currents $\mathcal{N} = 4$ SYM offers the possibility of studying operators with large R-charge, *i.e.* large angular momentum J on S^5. Decomposing $SO(6)$ under $U(1)_J \times SU(2) \times SU(2)$, it is easy to see that CPO's of the form $Tr(Z^J)$ with $\Delta = J$ are 1/2 BPS and protected since there are no other operators they can possibly mix with (in the planar limit). All other operators can be built by successively inserting *impurities* with $\Delta > J$. In particular four real scalars, four of the gaugini and the four derivatives have $\Delta - J = 1$. Berenstein, Maldacena and Nastase [49] argued that the sector of operators with large R-charge is dual to the type IIB superstring on the maximally supersymmetric pp-wave that emerges from a Penrose limit of $AdS_5 \times S^5$ [50, 51]. Despite the presence of a null RR 5-form flux $F_{+1234} = F_{+5678} = \mu$, superstring fluctuations around the resulting background can be quantized in the light-cone gauge [52, 53] whereby $p^+ = J/(\mu\alpha')$ and the 'vacuum' $|p^+\rangle$ corresponds to $Tr(Z^J)$. The spectrum of the light-cone Hamiltonian

$$H_{LC} = p^- = \mu(\Delta - J) = \mu \sum_n N_n \omega_n \quad , \quad \omega_n = \sqrt{1 + \frac{n^2\lambda}{J^2}} \quad , \tag{2.6}$$

combined with the level matching condition $\sum_n nN_n = 0$, thus gives a prediction for the anomalous dimensions of so-called BMN operators with large R-charge. BMN operators with $\Delta = J+1$ (one impurity) are necessarily superconformal descendants of the 'vacuum' and are thus protected. Operators with two impurities, say X and Y,

$$a^X_n a^Y_{-n} |p^+\rangle \quad \leftrightarrow \quad \sum_k e^{2\pi i k n/J} Tr(X Z^{J-k} Y Z^k) \tag{2.7}$$

are in general unprotected, since $\Delta = J + 2\omega_n > J + 2$ for $n \neq 0$. At large but finite J BMN operators form $(P)SU(2, 2|4)$ multiplets [54] whose structure becomes more and

more involved with the number of impurities. For our later purposes it is crucial that any single-trace operator in $\mathcal{N}=4$ can be identified with some component of a BMN multiplet with an arbitrary but finite number of impurities.

3 Superstring spectrum on stringy $AdS_5 \times S^5$

At vanishing coupling, free $\mathcal{N}=4$ SYM exposes HS Symmetry enhancement [42–45]. Conformal invariance indeed implies that a spin s current, such as

$$J_{(\mu_1\mu_2\ldots\mu_s)|} = Tr(\varphi_i D_{(\mu_1}\ldots D_{\mu_s)|}\varphi^i) + \ldots \quad , \tag{3.1}$$

saturating the unitary bound $\Delta_0 = 2+s$ be conserved. $\mathcal{N}=4$ superconformal symmetry implies that twist two operators are either conserved currents or superpartners thereof [55–57]. Once interactions are turned off long multiplets decompose into (semi)short ones forming the *doubleton* representation of $HS(2,2|4)$, the HS extension of $(P)SU(2,2|4)$.

The weak coupling regime on the boundary should be holographically dual to a highly stringy regime in the bulk, where the curvature radius R is small in string units $R \approx \sqrt{\alpha'}$ and the string is nearly tensionless [58, 59]. Although, as remarked above, quantizing the superstring in $AdS_5 \times S^5$ is a difficult and not yet accomplished task, the huge enhancement of symmetry allows us to determine the superstring spectrum in this limit and to precisely match it with the $\mathcal{N}=4$ SYM spectrum. To this end, it is first convenient to recall the structure of the type IIB superstring spectrum in flat spacetime.

In the light-cone GS formalism (left-moving) superstring excitations are obtained by acting on the groundstate $|\mathcal{Q}\rangle$ with the the $\mathbf{8}_V$ bosonic, α^I_{-n}, with $I = 1, \ldots 8$, and the the $\mathbf{8}_S$ fermionic, S^a_{-n}, with $a = 1, \ldots 8$, creation operators. As a result of the quantization of the fermionic zero-modes S^a_0 the groundstate $|\mathcal{Q}\rangle = |I\rangle - |\dot{a}\rangle$ is 16-fold degenerate and consists of $\mathbf{8}_V$ bosons and $\mathbf{8}_C$ fermions. Combining with right-moving modes with the same chirality projection on the vacuum and imposing level-matching $\ell = \sum_n n N^L_n = \sum_n n N^R_n$ one obtains the complete physical spectrum of 'transverse' single-particle excitations. At $\ell = 0$ one finds the components of the type IIB $\mathcal{N} = (2,0)$ supergravity multiplet: the graviton $G_{(IJ)|}$, two antisymmetric tensors $B^r_{[IJ]}$, two scalars ϕ^r and a four-index self-dual antisymmetric tensor A_{IJKL}, for a total of $35 + 2 \times 28 + 2 + 35 = 128$ bosonic d.o.f.; two gravitini Ψ^u_{Ia} and two dilatini $\Lambda^u_{\dot{a}}$, for a total of $2 \times (56 + 8) = 128$ fermionic d.o.f..

At higher levels, $\ell \geq 1$, the (chiral) spectrum not unexpectedly assembles into full representations of the massive transverse Lorentz group $SO(9)$. Indeed, focussing for simplicity on the left-moving sector at lowest level one finds

$$\ell = 1 \quad : \quad (\alpha^I_{-1} - S^a_{-1})(|J\rangle - |\dot{b}\rangle) \tag{3.2}$$

yielding

$$(\mathbf{8}_V - \mathbf{8}_S)(\mathbf{8}_V - \mathbf{8}_C) = (\mathbf{1}_O + \mathbf{28}_O + \mathbf{35}_O + \mathbf{8}_V + \mathbf{56}_V) - (\mathbf{8}_C + \mathbf{56}_C + \mathbf{8}_S + \mathbf{56}_S) \quad (3.3)$$

that can be reorganized into $SO(9)$ representations

$$\ell = 1 \quad : \quad \mathbf{44} + \mathbf{84} - \mathbf{128} \quad (3.4)$$

corresponding to a symmetric tensor ('spin 2'), a 3-index totally antisymmetric tensor and a spin-vector ('spin 3/2'). Incidentally this is exactly the (massless) field content of $D = 11$ supergravity. For $\ell > 1$ the situation is analogous though more involved.

The next task is to decompose the spectrum in $\mathcal{N} = (2, 0)$ supermultiplets, *i.e.* identify the groundstates annihilated by half of the $32 = 2 \times 16$ supercharges, that play the role of 'lowering' operators. For $\ell = 1$ the groundstate cannot be anything else than an $SO(9)$ singlet $V^{L/R}_{\ell=1} = \mathbf{1}$, *i.e.* a scalar, since $2^8 \times 2^8 = 2^{16}$ equals the number of d.o.f. at this level. At higher level the situation is not so straightforward. It proves convenient to 'factor out' – for both left- and right-movers – the structure $\mathcal{Q}_S \times \mathcal{Q}_C = (\mathbf{8}_V - \mathbf{8}_S)(\mathbf{8}_V - \mathbf{8}_C)$ that corresponds to the action of the 8 'raising' supercharges. Using this trick one can deduce a recurrence relation that yields

$$V^{L/R}_{\ell=1} = \mathbf{1} \quad , \quad V^{L/R}_{\ell=2} = \mathbf{9} \quad , \quad V^{L/R}_{\ell=3} = \mathbf{44} - \mathbf{16} \quad , \quad \ldots \quad (3.5)$$

for the first few levels. In summary, the Hilbert space of type IIB superstring excitations in flat space can be written as

$$\mathcal{H}_{flat} = \mathcal{H}_{sugra} + \mathcal{T}_{susy} \sum_{\ell} V^L_\ell \times V^R_\ell \quad (3.6)$$

where $\mathcal{T}_{susy} = [\mathcal{Q}_S \times \mathcal{Q}_C]_L \times [\mathcal{Q}_S \times \mathcal{Q}_C]_R$ represents the action of the supercharges.

Two amusing features are worth noticing at this point. First, the maximum spin at level ℓ is $s_{Max} = 2\ell + 2$. States with such spin and their superpartners are said to belong to the first Regge trajectory which is generated by the oscillators with lowest possible mode number. Second, the partial sums $\sum^{1,K}_{\ell} V^L_\ell \times V^R_\ell$ form $SO(10)$ multiplets. This can be related to the possibility of 'covariantizing' the massive spectrum of type IIB, which is identical to the one of type IIA, to $SO(10)$, by lifting it to $D = 11$ [60,61], or to $SO(9, 1)$, by introducing the worldsheet (super)ghosts.

In order to extrapolate the massive string spectrum from flat space to $AdS_5 \times S^5$ at the HS spin symmetry enhancement point one can take the following steps. First, decompose $SO(9)$ into $SO(4) \times SO(5)$, the relevant stability group of a massive particle; second, identify the KK towers of spherical harmonics that replace the (internal) momenta; third, assign an AdS mass M, dual to the boundary scaling dimension Δ, to each state.

The first step is completely straightforward and determines two of the quantum numbers of the relevant $PSU(2,2|4)$ superisometry group, namely the two spins (j_L, j_R) of $SO(4) \subset SO(4,2)$. In more covariant terms and to linear order in fluctuations around $AdS_5 \times S^5$, diagonalization of the type IIB field equations should produce a set of uncoupled free massive equations:

$$\left(\nabla^2_{AdS_5 \times S^5} - M^2_{\Phi}\right) \Phi_{\{\mu\}\,\{i\}} = 0 \tag{3.7}$$

The collective indices $\{\mu\} \in \mathcal{R}_{SO(1,4)}$ and $\{i\} \in \mathcal{R}_{SO(5)}$ label irreducible representations of the relevant Lorentz group, $SO(4,1) \times SO(5)$. To each excitation around flat spacetime we associate a tower of KK recurrences. The spectrum of AdS masses M_Φ taking into account the coupling to the curvature and RR 5-form flux should be determined by requiring consistency, *i.e.* BRS invariance, of superstring propagation on $AdS_5 \times S^5$. Each ten-dimensional field $\Phi_{\{\mu\}\,\{i\}}$ can be expanded in S^5-spherical harmonics:

$$\Phi_{\{\mu\}\,\{i\}}(x,y) = \sum_{[k,p,q]} \mathcal{X}^{[kpq]}_{\{\mu\}}(x)\, \mathcal{Y}^{[kpq]}_{\{i\}}(y) \ , \tag{3.8}$$

with x, y coordinates along AdS_5 and S^5 respectively. The sum runs over the set of allowed representations of the S^5 isometry group $SO(6) \approx SU(4)$ characterized by their $SU(4)$ Dynkin labels $[k,p,q]$ and the (generalized) spherical harmonics $\mathcal{Y}^{[kpq]}_{\{i\}}(y)$ are eigenvectors of the S^5 Laplacian:

$$\nabla^2_{S^5}\, \mathcal{Y}^{[kpq]}_{\{i\}} \;\; = \;\; -\left(C_2\left[SO(6)\right] - C_2\left[SO(5)\right]\right) \mathcal{Y}^{[kpq]}_{\{i\}} \tag{3.9}$$

with $C_2[G]$ standing for the second Casimir of the group G.

The second step thus corresponds to finding all the irreducible representations of $SO(6)$ that contain a given representation of $SO(5)$ under the decomposition $SO(6) \to SO(5)$. Denoting $SO(5)$ irreps by their Dynkin labels, $[m,n]$, standard group theory analysis yields the KK towers

$$\mathrm{KK}_{[m,n]} = \sum_{r=0}^{m} \sum_{s=0}^{n} \sum_{p=m-r}^{\infty} [r+s,\, p,\, r+n-s] + \sum_{r=0}^{m-1} \sum_{s=0}^{n-1} \sum_{p=m-r-1}^{\infty} [r+s+1,\, p,\, r+n-s] \ . \tag{3.10}$$

In particular for the smallest irreps one gets: $KK_{\mathbf{1}=[0,0]} = \sum_{p=0}^{\infty}[0,p,0]$, $KK_{\mathbf{5}=[1,0]} = \sum_{p=1}^{\infty}[0,p,0] + \sum_{p=0}^{\infty}[1,p,1]$, and $KK_{\mathbf{4}=[0,1]} = \sum_{n=0}^{\infty}[1,p,0] + [0,p,1]$. Notice that any ambiguity in the lift, say, of the (pseudo-real) spinor $\mathbf{4}$ of $SO(5) \approx Sp(4)$ to the complex $\mathbf{4}$ of $SO(6) \approx SU(4)$ or to its complex conjugate $\mathbf{4}^*$ is resolved by the infinite sum over KK recurrences. Once the $SO(6)$ R-symmetry quantum numbers are determined, the perturbative spectrum of the superstring turns out to be encoded in

$$\mathcal{H}_{AdS} = \mathcal{H}_{sugra} + \mathcal{T}_{KK} \mathcal{T}_{susy} \sum_{\ell} V^L_\ell \times V^R_\ell \tag{3.11}$$

where $\mathcal{T}_{KK} = \sum_p [0,p,0]$ represents the KK tower that, as indicated, boils down to a sum over scalar spherical harmonics. $\mathcal{T}_{susy}$ represents the action of the 16 'raising' supercharges Q and $\bar{Q}$ with quantum numbers $\{1/2; (1/2,0); [1,0,0]\}$ and $\{1/2; (0,1/2); [0,0,1]\}$, respectively. $V_\ell^{L/R}$, defined in flat space, are to be decomposed under $SO(4) \times SO(5)$. Formula (3.11) looks deceivingly simple, almost trivial, since the most interesting and subtle information, the scaling dimension Δ_0, at the HS enhancement point, is still missing.

Before addressing this crucial issue, two remarks are in order. First we have tacitly assumed that there are no non-perturbative states that can appear in the single-particle spectrum as a result of strings or branes wrapping non trivial cycles [16]. Indeed there are no such states with finite mass, since the only non trivial cycles of S^5 are a 0-cycle (a point) or a 5-cycle (the full space). Although there are no stable type IIB 0-branes there are stable 5-branes of various kinds. However they give rise to very massive objects (baryon vertices, ...) at small string coupling, *i.e.* large N. Second, there can be ambiguities in extrapolating the perturbative spectrum from large radius, where KK technology is reliable but string excitations are very massive, to small radius where HS symmetry is restored but stringy geometry should replace more familiar concepts. We should then appeal to the non-intersecting principle [62] that guarantees that any state identified at large radius (strong 't Hooft coupling) can be smoothly followed to weak coupling and viceversa. Indeed whenever the dimensions of two (or more) operators with the same quantum numbers start to approach one another level repulsion should prevent them from actually coincide.

One can then start by identifying the string excitations that are expected to become massless at the point of enhanced HS symmetry. In particular the totally symmetric and traceless tensors of rank $2\ell - 2$ at level $\ell > 1$ appearing in the product of the groundstates $V_\ell^L \times V_\ell^R$ become massless and thus correspond to the sought for conserved currents on the boundary if one assigns them $\Delta_0 = 2\ell$, that works fine for $\ell = 1$, too. The states with quantumm numbers $\{2\ell; (\ell-1, \ell-1); [0,0,0]\}$ are HWS's of semishort multiplets. $PSU(2,2|4)$ symmetry then fixes the scaling dimensions of the other components. In practice, *one takes* 2^{16} *birds with one stone!* Moreover the KK recurrences of these states at floor p arising from $\mathcal{T}_{KK}$ are naturally assigned

$$\Delta_0 = 2\ell + p \tag{3.12}$$

which represents the $PSU(2,2|4)$ unitary bound for a spin $s = 2\ell - 2$ current in the $SO(6)$ irrep with Dynkin labels $[0,p,0]$. It is remarkable how simply assuming HS symmetry enhancement fixes the AdS masses, *i.e.* scaling dimensions, of a significant fraction of the spectrum. Although experience with perturbative gauge theories teaches us that even at this particularly symmetric point there be operators / states well above the relevant

$PSU(2,2|4)$ unitary bounds, the above very simple yet effective mass formula turns out to be correct for all states with dimension up to $\Delta_0 = 4$. Notice that 'commensurability' of the two contribution – spin $s \approx \ell$ and KK 'angular momentum' $J \approx p$ – suggests $R = \sqrt{\alpha'}$, for what this could mean. In order to find a mass formula that could extend and generalize the above one, it is convenient to take the BMN formula as a hint. Although derived under the assumptions of large λ and J there seems to be no serious problem in extrapolating it to finite J at vanishing λ, where $\omega_n = 1$ for all n. Indeed (two-impurity) BMN operators form $PSU(2,2|4)$ multiplets at finite J and are thus amenable to the extrapolation [54]. The resulting formula reads

$$\Delta_0 = J + \nu \tag{3.13}$$

where $\nu = \sum_n N_n$ is the number of oscillators applied to the 'vacumm' $|J = \mu\alpha' p^+\rangle$ and J is a $U(1)$ R-charge yet to be identified. The easiest way to proceed is to first 'covariantize' $SO(9)$ to $SO(10)$ and then decompose the latter into $SO(8) \times U(1)_J$ where $SO(8)$ is the massless little group and $U(1)_J$ is precisely the sought for R-charge. Although cumbersome the procedure is straightforward and can be easily implemented on a computer. Given the $SO(10)$ content of the flat space string spectrum, equation (3.13) uniquely determines the dimensions Δ_0 of the superstring excitations around $AdS_5 \times S^5$ at the HS point. As an illustration, let us consider the first few string levels:

$$\begin{aligned} V_1 &= [0,0,0,0,0]^1 \\ &\overset{SO(8)\times SO(2)}{\longrightarrow} [0,0,0,0]^1_0 \\ &\overset{(3.13)}{\longrightarrow} [0,0,0,0]_1 \ , \end{aligned} \tag{3.14}$$

$$\begin{aligned} V_2 &= [1,0,0,0,0]^2 - [0,0,0,0,0]^3 \\ &\overset{SO(8)\times SO(2)}{\longrightarrow} [1,0,0,0]^2_0 + [0,0,0,0]^2_1 + [0,0,0,0]^2_{-1} - [0,0,0,0]^3_0 \qquad (3.15) \\ &\overset{(3.13)}{\longrightarrow} [1,0,0,0]_2 + [0,0,0,0]_1 \ , \qquad (3.16) \end{aligned}$$

$$\begin{aligned} V_3 &= [2,0,0,0,0]^3 - [1,0,0,0,0]^4 - [0,0,0,0,1]^{5/2} \\ &\overset{SO(8)\times SO(2)}{\longrightarrow} [2,0,0,0]^3_0 + [1,0,0,0]^3_1 + [1,0,0,0]^3_{-1} + [0,0,0,0]^3_0 + [0,0,0,0]^3_2 \\ &\quad + [0,0,0,0]^3_{-2} - [1,0,0,0]^4_0 - [0,0,0,0]^4_1 - [0,0,0,0]^4_{-1} \\ &\quad - [0,0,0,1]^{5/2}_{1/2} - [0,0,1,0]^{5/2}_{-1/2} \\ &\overset{(3.13)}{\longrightarrow} [2,0,0,0]_3 + [1,0,0,0]_2 + [0,0,0,0]_1 - [0,0,1,0]_3 - [0,0,0,1]_3 \end{aligned} \tag{3.17}$$

With the above assignments of Δ_0, negative multiplicities are harmless since they cancel in the sum over KK recurrences after decomposing $SO(10)$ w.r.t. $SO(4) \times SO(6)$. For these low massive levels, the conformal dimensions determined by (3.13) all saturate $SO(10)$ unitary bounds $\Delta_\pm = 1+k+2l+3m+2(p+q)\pm(p-q)/2$. At higher levels, starting from a scalar singlet with $\Delta_0 = 3$ at level $\ell = 5$, these bounds are satisfied but no longer saturated. The correct conformal dimensions are rather obtained from (3.13). Notice that the first fermionic primary appears at level $\ell = 3$ and has dimension $\Delta_0 = 11/2$.

Summarizing, the massive flat space string spectrum may be lifted to $SO(10)\times SO(2)_\Delta$, such that breaking $SO(10)$ down to $SO(8) \times SO(2)_J$ reproduces the original $SO(8)$ string spectrum and its excitation numbers via the relation (3.13). The results up to string level $\ell = 5$ are displayed in the following tables and organized under $SO(10) \times SO(2)_\Delta$, with Dynkin labels $[k, l, m, p, q]_{\Delta_0}$ and $[k, l, m, p, q]^* \equiv [k, l, m, p, q] - [k-1, l, m, p, q]$.

$\ell = 1$:

Δ_0	$\mathcal{R}$
1	$[0,0,0,0,0]$

$\ell = 2$:

Δ_0	$\mathcal{R}$
2	$[1,0,0,0,0]^*$

$\ell = 3$:

Δ_0	$\mathcal{R}$
3	$[2,0,0,0,0]^*$
$\frac{5}{2}$	$[0,0,0,0,1]$

$\ell = 4$:

Δ_0	$\mathcal{R}$
4	$[3,0,0,0,0]^*$
$\frac{7}{2}$	$[1,0,0,0,1]^*$
3	$[0,1,0,0,0]$

$\ell = 5$:

Δ_0	$\mathcal{R}$
5	$[4,0,0,0,0]^*$
$\frac{9}{2}$	$[2,0,0,0,1]^*$
4	$[0,0,1,0,0] + [1,1,0,0,0]^*$
$\frac{7}{2}$	$[1,0,0,0,1]$
3	$[0,0,0,0,0]$

4 The spectrum of free $\mathcal{N} = 4$ SYM

In order to test the above prediction for the single-particle superstring spectrum on $AdS_5 \times S^5$ at the HS point with the spectrum of free $\mathcal{N} = 4$ SYM theory at large N, one has to devise an efficient way of computing gauge-invariant single trace operators. For $SU(N)$ gauge group this means taking care of the ciclicity of the trace in order to avoid multiple counting [63, 43, 64, 62, 1, 65]. Moreover one should discard operators which would vanish along the solutions of the field equations and deal with the statistics of the elementary fields properly. The mathematical tool one has to resort to is Polya theorem [66] that allows one to count 'words' $A, B, ...$ of a given 'length' n composed of 'letters' chosen from a given alphabet $\{a_i\}$, modulo some symmetry operation: $A \approx B$ if $A = gB$ for $g \in \mathcal{G}$. Decomposing the discrete group $\mathcal{G} \subset \mathcal{S}_n$ into conjugacy classes whose representatives $[g] = (1)^{b_1(g)}(1)^{b_1(g)}...(n)^{b_n(g)}$ are characterized by the numbers $b_k(g)$ of cycles of length k, Polya cycle index is given by

$$\mathcal{P}_{\mathcal{G}}(\{a_i\}) = \frac{1}{|\mathcal{G}|}\sum_g \prod_{k=1}^{n} (\sum_i a_i^k)^{b_k(g)} \tag{4.1}$$

For cyclic groups, $\mathcal{G} = Z_n$, conjugacy classes are labeled by divisors d of n, $[g]_d = (d)^{n/d}$, and the cycle index simply reads

$$\mathcal{P}_{Z_n}(\{a_i\}) = \frac{1}{n}\sum_{d|n} \mathcal{E}(d)(\sum_i a_i^d)^{n/d} \tag{4.2}$$

where $\mathcal{E}(d)$ is Euler totient function which counts the number of elements in the conjugacy class $[g]_d$. $\mathcal{E}(d)$ equals the number of integers relatively prime to and smaller than d, with the understanding that $\mathcal{E}(1) = 1$, and satisfies $\sum_{d|n} \mathcal{E}(d) = n$.

For $\mathcal{N} = 4$ SYM the alphabet consists of the elementary fields together with their derivatives $\{\partial^k\varphi, \partial^k\lambda, \partial^k F\}$, modulo the field equations, that transform in the *singleton* representation of $PSU(2,2|4)$. As a first step, one computes the on-shell single letter partition function $\mathcal{Z}_1(q) = Trq^{\Delta_0}$, where q keeps track of the 'naive' scaling dimension Δ_0[4]. For a free scalar of dimension $\Delta_0 = 1$ in $D = 4$

$$\mathcal{Z}_1^{(s)}(q) = q\frac{1-q^2}{(1-q)^4} = q\frac{1+q}{(1-q)^3} \tag{4.3}$$

where $-q^2$ removes the (module of the) null descendant $\partial^2\varphi = 0$. For a free Weyl fermion of dimension $\Delta_0 = 3/2$

$$\mathcal{Z}_1^{(f)}(q) = 2q^{3/2}\frac{1-q}{(1-q)^4} = 2q^{3/2}\frac{1}{(1-q)^3} \tag{4.4}$$

[4] Additional variables can introduced in order to keep track of other quantum numbers and compute the character valued partition function.

where $-q$ removes the null descendant $\partial\!\!\!/\lambda = 0$. For a free vector field or rather its field strength of dimension $\Delta_0 = 2$

$$\mathcal{Z}_1^{(v)}(q) = 2q^2\frac{(3-4q+q^2)}{(1-q)^4} = 2q^2\frac{3-q}{(1-q)^3} \tag{4.5}$$

where $-4q$ removes the null descendants at level one $\partial_\mu F^{\mu\nu} = \partial_\mu \widetilde{F}^{\mu\nu} = 0$ and $+q^2$ takes care of the algebraic identities $\partial_\mu\partial_\nu F^{\mu\nu} = \partial_\mu\partial_\nu \widetilde{F}^{\mu\nu} = 0$ at the next level.

Taking into account statistics, *i.e.* computing the Witten index $Tr(-)^F q^\Delta$, and setting $n_s = 6$, $n_f = n_{\bar{f}} = 4$ and $n_v = 1$ one gets

$$\mathcal{Z}_1^{(\mathcal{N}=4)}(q) = 2q\frac{(3+\sqrt{q})}{(1+\sqrt{q})^3} \tag{4.6}$$

for a single (abelian) $\mathcal{N}=4$ vector multiplet, quite remarkably $\mathcal{Z}_1^{(\mathcal{N}=4)}(q) = \mathcal{Z}_1^{(v)}(-\sqrt{q})$.

Plugging this in (4.2) and removing the single-letter term ($n=1$), as appropriate for an $SU(N)$ gauge group, one finds

$$\begin{aligned}
\mathcal{Z}^{(\mathcal{N}=4)}(q) &= \sum_{n=2}^{\infty}\sum_{n|d}\frac{\mathcal{E}(d)}{n}\left[\frac{2q(3+q^{\frac{d}{2}})}{(1+q^{\frac{d}{2}})^3}\right]^{\frac{n}{d}} && (4.7)\\
&= 21\,q^2 - 96\,q^{\frac{5}{2}} + 376\,q^3 - 1344\,q^{\frac{7}{2}} + 4605\,q^4 - 15456\,q^{\frac{9}{2}} + 52152\,q^5 \\
&\quad - 177600\,q^{\frac{11}{2}} + 608365\,q^6 - 2095584\,q^{\frac{13}{2}} + 7262256\,q^7 - 25299744\,q^{\frac{15}{2}} \\
&\quad + 88521741\,q^8 - 310927104\,q^{\frac{17}{2}} + 1095923200\,q^9 - 3874803840\,q^{\frac{19}{2}} \\
&\quad + 13737944493\,q^{10} + \mathcal{O}(q^{\frac{21}{2}}) \quad . && (4.8)
\end{aligned}$$

At large N mixing with multi-trace operators is suppressed.

The next step is to identify super-primaries, which is tantamount to passing $\mathcal{Z}^{(\mathcal{N}=4)}(q)$ through an Eratosthenes super-sieve, that removes super-descendants. This task can be accomplished by first subtracting the contribution of 1/2 BPS multiplets from (4.7)

$$\begin{aligned}
\mathcal{Z}_{BPS}^{(\mathcal{N}=4)}(q) &= \sum_{n=2}^{\infty}\frac{q^n\big(n+2-(n-2)q^{\frac{1}{2}}\big)}{12(1+q^{\frac{1}{2}})^4}\Big[(n+1)(n+3)\big((n+2)-3q^{\frac{1}{2}}(n-2)\big) + \\
&\qquad + q(n-1)(n-3)\big(3(n+2)-q^{\frac{1}{2}}(n-2)\big)\Big] \\
&= \frac{q^2\big(20+80q^{\frac{1}{2}}+146q+144q^{\frac{3}{2}}+81q^2+24q^{\frac{5}{2}}+3q^3\big)}{(1-q)(1+q^{\frac{1}{2}})^8}\,, && (4.9)
\end{aligned}$$

from (4.7) and then dividing by

$$\mathcal{T}_{SO(10,2)}(q) = (1-q^2)\frac{(1-q^{\frac{1}{2}})^{16}}{(1-q)^{10}}. \tag{4.10}$$

that not only removes superconformal descendants generated by

$$\mathcal{T}_{susy}(q) = \frac{(1-q^{\frac{1}{2}})^{16}}{(1-q)^4}. \tag{4.11}$$

but also the operators dual to the KK recurrences generated by

$$\mathcal{T}_{KK}(q) = \frac{(1-q^2)}{(1-q)^6}. \tag{4.12}$$

where the numerator implements the $SO(6)$ tracelessness condition.

One eventually finds

$$\begin{aligned}
\mathcal{Z}^{(\mathcal{N}=4)}_{SO(10,2)}(q) &= \left[\mathcal{Z}^{(\mathcal{N}=4)}(q) - \mathcal{Z}^{(\mathcal{N}=4)}_{BPS}(q)\right]/\mathcal{T}_{SO(10,2)}(q) \qquad (4.13)\\
&= q^2 + 100\,q^4 + 236\,q^5 - 1728\,q^{\frac{11}{2}} + 4943\,q^6 - 12928\,q^{\frac{13}{2}}\\
&\quad + 60428\,q^7 - 201792\,q^{\frac{15}{2}} + 707426\,q^8 - 2550208\,q^{\frac{17}{2}}\\
&\quad + 9101288\,q^9 - 32568832\,q^{\frac{19}{2}} + 116831861\,q^{10} + \mathcal{O}(q^{\frac{21}{2}})\ .
\end{aligned}$$

The expansion (4.13) can be reorganized in the form

$$\begin{aligned}
\mathcal{Z}^{(\mathcal{N}=4)}_{SO(10,2)}(q) &= (q^1)^2 + (10q^2 - q^3)^2 + (-16q^{5/2} + 54q^3 - 10q^4)^2\\
&\quad + (45q^3 - 144q^{\frac{7}{2}} + 210q^4 + 16q^{\frac{9}{2}} - 54q^5)^2 + \dots\ , \qquad (4.14)
\end{aligned}$$

It is not difficult to recognize that

$$\mathcal{Z}^{(\mathcal{N}=4)}_{SO(10,2)}(q) = \sum_{\ell} V^L_\ell(q) \times V^R_\ell(q) \tag{4.15}$$

where q keeps track of the dimensions assigned via $\Delta_0 = J + \nu$ after lifting $SO(9)$ to $SO(10)$. We thus find perfect agreement with the previously derived string spectrum up to $\Delta_0 = 10$, and are confident that our assumptions and extrapolations to the HS enhancement point are correct. The final result seems to suggest holomorphic factorization of the string worldsheet dynamics at this particularly symmetric point. The origin of the $SO(10,2)$ spectrum symmetry calls for deeper understanding possibly in connection with Bars's two-time formulation of the type IIB superstring [67,61]. In the following table we gather the spectrum of super-primaries of long multiplets in $\mathcal{N}=4$ SYM. For reasons of limited space, we present the result only up to $\Delta_0 = 13/2$. HWS of long multiplets of $\mathcal{N}=4$ are denoted by $[j_L, j_R; k, p, q]^P_{L,B}$, where (j_L, j_R) and $[k, p, q]$ indicate the Dynkin labels of $SO(4)$ and $SO(6)$, respectively. In addition we include parity P, described in [69], and hypercharge (Intriligator's 'bonus' symmetry) B, the leading order $U(1)_B$ charge in the decomposition $SU(2,2|4) = U(1)_B \times PSU(2,2|4)$, and length L, the leading order

number of letters / partons. Furthermore, $P = \pm$ indicates a pair of states with opposite parities while +conj. indicates a conjugate state $[j_R, j_L; q, p, k]^P_{L,-B}$.

Δ_0	$\mathcal{R}$
2	$[0,0;0,0,0]^+_{2,0}$
3	$[0,0;0,1,0]^-_{3,0}$
4	$2\cdot[4;0,0;0,0,0]^+_{4,0} + [0,0;1,0,1]^-_{4,0} + 2\cdot[4;0,0;0,2,0]^+_{4,0}$ $+ ([0,2;0,0,0]^-_{3,-1} + \text{conj.}) + [1,1;0,1,0]^\pm_{3,0} + [2,2;0,0,0]^+_{2,0}$
5	$4\cdot[0,0;0,1,0]^-_{5,0} + 2\cdot([0,0;0,0,2]^+_{5,0} + \text{conj.}) + [0,0;1,1,1]^\pm_{5,0}$ $+ 2\cdot[5;0,0;0,3,0]^-_{5,0} + 2\cdot([0,2;0,1,0]^+_{4,-1} + \text{conj.})$ $+ ([0,2;2,0,0]^-_{4,-1} + \text{conj.}) + [1,1;0,0,0]^\pm_{4,0} + 2\cdot[1,1;1,0,1]^\pm_{4,0}$ $+ [1,1;0,2,0]^\pm_{4,0} + [2,2;0,1,0]^-_{3,0}$
$\frac{11}{2}$	$2\cdot[0,1;1,0,0]^\pm_{5,-1/2} + 2\cdot[0,1;0,1,1]^\pm_{5,-1/2} + [0,1;1,2,0]^\pm_{5,-1/2}$ $+ 2\cdot[1,2;0,0,1]^\pm_{4,-1/2} + [1,2;1,1,0]^\pm_{4,-1/2} + [2,3;1,0,0]^\pm_{3,-1/2} + \text{conjugates}$
6	$2\cdot[0,0;0,0,0]^+_{4,0} + 2\cdot([0,0;0,0,0]^+_{5,-1} + \text{conj.}) + 5\cdot[0,0;0,0,0]^+_{6,0} + 3\cdot[0,0;1,0,1]^+_{6,0}$ $+ 6\cdot[0,0;1,0,1]^-_{6,0} + 9\cdot[0,0;0,2,0]^+_{6,0} + [0,0;0,2,0]^-_{6,0} + 3\cdot([0,0;0,1,2]^-_{6,0} + \text{conj.})$ $+ 3\cdot[0,0;2,0,2]^+_{6,0} + [0,0;1,2,1]^+_{6,0} + 2\cdot[0,0;1,2,1]^-_{6,0} + 3\cdot[0,0;0,4,0]^+_{6,0}$ $+ 2\cdot([0,2;0,0,0]^-_{4,0} + \text{conj.}) + 3\cdot([0,2;0,0,0]^-_{5,-1} + \text{conj.})$ $+ 4\cdot([0,2;1,0,1]^+_{5,-1} + \text{conj.}) + 2\cdot([0,2;1,0,1]^-_{5,-1} + \text{conj.})$ $+ 4\cdot([0,2;0,2,0]^-_{5,-1} + \text{conj.}) + ([0,2;2,1,0]^+_{5,-1} + \text{conj.}) + 8\cdot[1,1;0,1,0]^\pm_{5,0}$ $+ 2\cdot([1,1;0,0,2]^\pm_{5,0} + \text{conj.}) + 4\cdot[1,1;1,1,1]^\pm_{5,0} + 2\cdot[1,1;0,3,0]^\pm_{5,0}$ $+ ([0,4;0,0,0]^+_{3,-1} + \text{conj.}) + ([0,4;0,0,0]^+_{4,-2} + \text{conj.}) + 2\cdot([1,3;0,1,0]^\pm_{4,-1} + \text{conj.})$ $+ 5\cdot[2,2;0,0,0]^+_{4,0} + 2\cdot[2,2;0,0,0]^-_{4,0} + 2\cdot[2,2;1,0,1]^-_{4,0} + 4\cdot[2,2;0,2,0]^+_{4,0}$ $+ [2,2;0,2,0]^-_{4,0} + ([2,4;0,0,0]^-_{3,-1} + \text{conj.}) + [3,3;0,1,0]^\pm_{3,0} + [4,4;0,0,0]^+_{2,0}$
$\frac{13}{2}$	$4\cdot[0,1;0,0,1]^\pm_{5,+1/2} + 6\cdot[0,1;0,0,1]^\pm_{6,-1/2} + 12\cdot[0,1;1,1,0]^\pm_{6,-1/2}$ $+ 5\cdot[0,1;1,0,2]^\pm_{6,-1/2} + [0,1;3,0,0]^\pm_{6,-1/2} + 5\cdot[0,1;0,2,1]^\pm_{6,-1/2}$ $+ 2\cdot[0,1;2,1,1]^\pm_{6,-1/2} + [0,1;1,3,0]^\pm_{6,-1/2} + [0,3;0,0,1]^\pm_{4,-1/2}$ $+ [0,3;0,0,1]^\pm_{5,-3/2} + 2\cdot[0,3;1,1,0]^\pm_{5,-3/2} + 10\cdot[1,2;1,0,0]^\pm_{5,-1/2}$ $+ 8\cdot[1,2;0,1,1]^\pm_{5,-1/2} + 3\cdot[1,2;2,0,1]^\pm_{5,-1/2} + 2\cdot[1,2;1,2,0]^\pm_{5,-1/2}$ $+ [1,4;1,0,0]^\pm_{4,-3/2} + 3\cdot[2,3;0,0,1]^\pm_{4,-1/2} + 2\cdot[2,3;1,1,0]^\pm_{4,-1/2} + \text{conjugates}$

5 HS symmetry and multiplets

It is now time to decompose the spectrum of single-trace operators in free $\mathcal{N} = 4$ SYM at large N or, equivalently, of type IIB superstring on $AdS_5 \times S^5$ extrapolated to the point of HS symmetry, into HS multiplets in order to set the stage for interactions and symmetry breaking. To this end we need to recall some basic properties of the infinite dimensional HS (super)algebra $\mathfrak{hs}(2,2|4)$, that extends the $\mathcal{N} = 4$ superconformal algebra $\mathfrak{psu}(2,2|4)$ [45, 70–72, 44, 73, 74]. The latter can be realized in terms of (super-)oscillators

$\zeta_\Lambda = (y_a, \theta_A)$ with:

$$[y_a, \bar{y}^b] = \delta_a{}^b \quad , \qquad \{\theta_A, \bar{\theta}^B\} = \delta^B{}_A \quad , \tag{5.1}$$

where $y_a, \bar{y}^b$ are bosonic oscillators with $a, b = 1, ...4$ a Weyl spinor index of $\mathfrak{so}(4,2) \sim \mathfrak{su}(2,2)$ or, equivalently, a Dirac spinor index of $\mathfrak{so}(4,1)$, while $\theta_A, \bar{\theta}^B$ are fermionic oscillators with $A, B = 1, ...4$ a Weyl spinor index of $\mathfrak{so}(6) \sim \mathfrak{su}(4)$.

Generators of $\mathfrak{psu}(2,2|4)$ are 'traceless' bilinears $\bar{\zeta}^\Sigma \zeta_\Lambda$ of superoscillators. In particular, the 'diagonal' combinations realize the compact (R-symmetry) $\mathfrak{so}(6)$ and noncompact (conformal) $\mathfrak{so}(4,2)$ bosonic subalgebras respectively, while the mixed combinations generate supersymmetries:

$$\begin{aligned} &J^a{}_b = \bar{y}^a y_b - \tfrac{1}{2} K \delta^a{}_b \quad , \quad K = \tfrac{1}{2} \bar{y}^a y_a \quad , \quad \bar{\mathcal{Q}}^a_A = \bar{y}^a \theta_A \quad , \\ &T^A{}_B = \bar{\theta}^A \theta_B - \tfrac{1}{2} B \delta^A_B \quad , \quad B = \tfrac{1}{2} \bar{\theta}^A \theta_A \quad , \quad \mathcal{Q}^A_a = \bar{\theta}^A y_a \quad . \end{aligned} \tag{5.2}$$

The central element

$$C \equiv K + B = \tfrac{1}{2} \bar{\zeta}^\Lambda \zeta_\Lambda \ , \tag{5.3}$$

generates an abelian ideal that can be modded out *e.g.* by consistently assigning $C = 0$ to the elementary SYM fields and their (perturbative) composites. Finally, the hypercharge B is the generator of Intriligator's 'bonus symmetry' [75, 76] dual to the 'anomalous' $U(1)_B$ chiral symmetry of type IIB supergravity. It acts as an external automorphism that rotates the supercharges.

The HS extension $\mathfrak{hs}(2,2|4)$ is roughly speaking generated by odd powers of the above generators *i.e.* combinations with equal odd numbers of ζ_Λ and $\bar{\zeta}^\Lambda$. More precisely, one first considers the enveloping algebra of $\mathfrak{psu}(2,2|4)$, which is an associative algebra and consists of all powers of the generators, then restricts it to the odd part which closes as a Lie algebra modulo the central charge C, and finally quotients the ideal generated by C. It is easy to show that B is never generated in commutators (but C is!) and thus remains an external automorphism of $\mathfrak{hs}(2,2|4)$. Generators of $\mathfrak{hs}(2,2|4)$ can be represented by 'traceless' polynomials in the superoscillators:

$$\mathfrak{hs}(2,2|4) \quad = \quad \oplus_\ell \mathcal{A}_{2\ell+1} = \sum_{\ell=0}^{\infty} \left\{ \mathcal{J}_{2\ell+1} = P^{\Lambda_1 \ldots \Lambda_{2\ell+1}}_{\Sigma_1 \ldots \Sigma_{2\ell+1}} \, \bar{\zeta}^{\Sigma_1} \ldots \bar{\zeta}^{\Sigma_{2\ell+1}} \, \zeta_{\Lambda_1} \ldots \zeta_{\Lambda_{2\ell+1}} \right\} \ , \tag{5.4}$$

with elements $\mathcal{J}_{2\ell+1}$ in $\mathcal{A}_{2\ell+1}$ at level ℓ parametrized by traceless rank $(2\ell+1)$ (graded) symmetric tensors $P^{\Lambda_1 \ldots \Lambda_{2\ell+1}}_{\Sigma_1 \ldots \Sigma_{2\ell+1}}$. Alternatively, the HS algebra can be defined by identifying generators differing by terms that involve C, i.e. $\mathcal{J} \approx \mathcal{K}$ iff $\mathcal{J} - \mathcal{K} = \sum_{k \geq 1} C^k \mathcal{H}_k$.

To each element in $hs(2,2|4)$ with $\mathfrak{su}(2)_L \times \mathfrak{su}(2)_R$ spins (j_L, j_R) is associated an $\mathfrak{hs}(2,2|4)$ HS currents and a dual HS gauge field in the AdS bulk with spins $(j_L+\frac{1}{2}, j_R+\frac{1}{2})$. The $\mathfrak{psu}(2,2|4)$ quantum numbers of the HS generators can be read off from (5.4) by expanding the polynomials in powers of θ's up to 4, since $\theta^5 = 0$. There is a single superconformal multiplet $\mathcal{V}_{2\ell}$ at each level $\ell \geq 2$. The lowest spin cases $\ell = 0, 1$, i.e. $\hat{\mathcal{V}}_{0,2}$, are special. They differ from the content of doubleton multiplets $\mathcal{V}_{0,2}$ by spin $s = 0, 1/2$ states [45]. The content of (5.4) can then be written as (tables 4,5 of [45])

$$\begin{aligned}
\hat{\mathcal{V}}_0 &= \left|\bar{\mathbf{4}}_{[\frac{1}{2},0]} + \mathbf{1}_{[1,0]}\right|^2 - \mathbf{1}_{[\frac{1}{2},\frac{1}{2}]} \\
\hat{\mathcal{V}}_2 &= \left|\mathbf{4}_{[\frac{1}{2},0]} + \mathbf{6}_{[1,0]} + \bar{\mathbf{4}}_{[\frac{3}{2},0]} + \mathbf{1}_{[2,0]}\right|^2 \\
\mathcal{V}_{2\ell} &= \left|\mathbf{1}_{[\ell-1,0]} + \mathbf{4}_{[\ell-\frac{1}{2},0]} + \mathbf{6}_{[\ell,0]} + \bar{\mathbf{4}}_{[\ell+\frac{1}{2},0]} + \mathbf{1}_{[\ell+1,0]}\right|^2 , \qquad \ell \geq 2 ,
\end{aligned} \tag{5.5}$$

with $\mathbf{r}_{[j_L+\frac{1}{2}, j_R+\frac{1}{2}]}$ denoting the $\mathfrak{su}(4)$ representation $\mathbf{r}$ and the $\mathfrak{su}(2)^2$ spins of the HWS's. Complex conjugates are given by conjugating $\mathfrak{su}(4)$ representations and exchanging the spins $j_L \leftrightarrow j_R$. The product is understood in $\mathfrak{su}(4)$ while spins simply add. The highest spin state $\mathbf{1}_{[\ell+1,\ell+1]}$ corresponds to the state $y^{2\ell+1}\bar{y}^{2\ell+1}$ with no θ's, $\mathbf{4}_{[\ell+\frac{1}{2},\ell+1]}$, $\bar{\mathbf{4}}_{[\ell+1,\ell+\frac{1}{2}]}$ to $y^{2\ell}\bar{y}^{2\ell+1}\theta^A$, $y^{2\ell+1}\bar{y}^{2\ell}\bar{\theta}_A$, and so on. For $\ell = 0, 1$, states with negative j_L, j_R should be deleted. In addition we subtract the current $\mathbf{1}_{[\frac{1}{2},\frac{1}{2}]}$ at $\ell = 0$ associated to C. In the $\mathcal{N} = 4$ notation introduced in Appendix A, $\mathcal{V}_{2\ell}$ corresponds to the semishort multiplet $\mathcal{V}^{2\ell,0}_{[000][\ell-1^*,\ell-1^*]}$.

The singleton representation $\mathcal{V}^{1,0}_{[0,1,0][0,0]}$ of $\mathfrak{psu}(2,2|4)$ truns out to be the fundamental representation of $\mathfrak{hs}(2,2|4)$, too. Its HWS $|Z\rangle$, or simply Z *i.e.* the ground-state or 'vacuum', which is not to be confused with the trivial $\mathfrak{psu}(2,2|4)$ invariant vacuum $|0\rangle$, is one of the complex scalars, say, $Z = \varphi^5 + i\varphi^6$. Showing that the singleton is an irreducible representation of $\mathfrak{psu}(2,2|4)$ is tantamount to showing that any state A in this representation can be found by acting on the vacuum Z, or any other state B, with a sequence of superconformal generators (5.2). Looking at the singleton as an irrep of $\mathfrak{hs}(2,2|4)$ one sees an important difference: the sequence of superconformal generators [5] is replaced by a single HS generator $\mathcal{J}_{A\bar{B}}$. Therefore any A in the $\mathfrak{hs}(2,2|4)$ singleton multiplet can be reached in a single step from any other one B as can be shown by noticing that, since the central charge C commutes with all generators and annihilates the vacuum Z, a non-trivial sequence in $(\mathcal{A}_1)^{2\ell+1}$ belongs to $\mathcal{A}_{2\ell+1}$. This property will be crucial in proving the irreducibility of YT-pletons with respect to the HS algebra. Let us then consider the tensor product of L singletons, *i.e.* L sites or partons. The generators

[5] Without loss of generality we may assume the length of the sequence to be odd; for an even sequence we may append an element of the Cartan subalgebra, e.g. the dilatation generator.

of $\mathfrak{hs}(2,2|4)$ are realized as diagonal combinations:

$$\mathcal{J}_{2\ell+1} \equiv \sum_{s=1}^{L} \mathcal{J}_{2\ell+1}^{(s)} \tag{5.6}$$

with $\mathcal{J}_{2\ell+1}^{(s)}$ HS generators acting on the s^{th} site. The tensor product of $L \geq 1$ singletons is generically reducible not only under $\mathfrak{psu}(2,2|4)$ but also under $\mathfrak{hs}(2,2|4)$. This can be seen by noticing that the HS generators (5.6), being completely symmetric, commute with (anti)symmetrizations of the indices. In particular, the tensor product decomposes into a sum of representations characterized by Young tableaux YT with L boxes.

To prove irreducibility of L-pletons associated to a specific YT under $\mathfrak{hs}(2,2|4)$, it is enough to show that any state in the L-pleton under consideration can be found by acting on the relevant HWS with HS generators. Let us start by considering states belonging to the totally symmetric tableau. The simplest examples of such states are those with only one 'impurity' i.e. $AZ \dots Z + \text{symm.}$. Using the fact that any SYM letter A can be reached from the HWS Z by means of a single $\mathfrak{hs}(2,2|4)$ generator $\mathcal{J}_{A\bar{Z}}$ we have $(\mathcal{J}_{A\bar{Z}}Z)Z \dots Z + \text{symm.} \approx \mathcal{J}_{A\bar{Z}}(Z^L)$ that shows the state is a HS descendant. The next simplest class is given by states with two impurities $ABZ \dots Z + \text{symm.}$. Once again these states can be written as $\mathcal{J}_{A\bar{Z}}\mathcal{J}_{B\bar{Z}}(Z^L)$, up to the one impurity descendants $(\mathcal{J}_{A\bar{Z}}\mathcal{J}_{B\bar{Z}}Z)Z \dots Z$ of the type already found. Proceeding in this way one can show that all states in the completely symmetric tensor of L singletons can be written as HS descendants of the vacuum Z^L.

The same arguments hold for generic tableaux. For example, besides the descendants $\mathcal{J}_{A\bar{Z}}(Z^L)$ of Z^L there are $L-1$ "one impurity" multiplets of states associated to the $L-1$ Young tableaux with $L-1$ boxes in the first row and a single box in the second one[6]. The vacuum state of HS multiplets associated to such tableaux can be taken to be $Y_{(k)} \equiv Z^k Y Z^{L-k-1} - YZ^{L-1}$ with $k = 1, \dots, L-1$, where $Y = \varphi^3 + i\varphi^4$ is another (complex) scalar. Any state with one impurity $Z^k A Z^{L-k-1} - AZ^{L-1}$ with $k = 1, \dots, L-1$ can be found by acting on $Y_{(k)}$ with the HS generator $\mathcal{J}_{A\bar{Y}}$, where $\mathcal{J}_{A\bar{Y}}$ is the HS generator that transforms Y into A (and annihilates Z). Notice that the arguments rely heavily on the fact that any two states in the singleton are related by a one-step action of a HS generator. This is not the case for the $\mathcal{N} = 4$ SCA, and indeed the completely symmetric tensor product of L singletons is highly reducible with respect to $\mathfrak{psu}(2,2|4)$.

The on-shell field content of the singleton representation of $\mathfrak{psu}(2,2|4)$ is encoded in the single-letter partition function $\mathcal{Z}_1(q) = \mathcal{Z}_{\square}(q)$. As previously described, the spectrum of single-trace operators in $\mathcal{N} = 4$ SYM theory with $SU(N)$ gauge group at large N can

[6]As we will momentarily see, HS multiplets of this kind are absent for $\mathcal{N} = 4$ SYM theories with semisimple gauge group. At any rate, they are instrumental to illustrate our point.

be extracted from the generating function [63, 43, 64, 62, 1, 65]

$$\mathcal{Z}(q) = \sum_{n\geq 2} \mathcal{Z}_n(q) = \sum_{n\geq 2, d|n} \frac{\mathcal{E}(d)}{n}\, \mathcal{Z}_{\ydiagram{1}}(q^d)^{\frac{n}{d}} \ , \tag{5.7}$$

of cyclic words of length $L = n$. Observe that $\mathcal{Z}_{\ydiagram{1}}(q^d)$ can be rewritten as the alternating sum over length-d Young tableaux of hook type:

$$\mathcal{Z}_{\ydiagram{1}}(q^d) \ = \ \mathcal{Z}_{\ydiagram{5}\cdot\cdot\ydiagram{2}}(q) \ - \ \mathcal{Z}_{\ydiagram{4,1}\cdot\cdot\ydiagram{2}}(q) \ + \ \mathcal{Z}_{\ydiagram{3,1,1}\cdot\cdot\ydiagram{2}}(q) \ - \ \mathcal{Z}_{\ydiagram{2,1,1,1}\cdot\cdot\ydiagram{2}}(q) \ + \ \dots \ . \tag{5.8}$$

Plugging this expansion into (5.7), we find for the first few cases:

$$\begin{aligned}
\mathcal{Z}_2 \ &= \ \mathcal{Z}_{\ydiagram{2}} \ , \\
\mathcal{Z}_3 \ &= \ \mathcal{Z}_{\ydiagram{3}} + \mathcal{Z}_{\ydiagram{1,1,1}} \ , \\
\mathcal{Z}_4 \ &= \ \mathcal{Z}_{\ydiagram{4}} + \mathcal{Z}_{\ydiagram{2,1,1}} + \mathcal{Z}_{\ydiagram{2,2}} \ , \\
\mathcal{Z}_5 \ &= \ \mathcal{Z}_{\ydiagram{5}} + \mathcal{Z}_{\ydiagram{3,2}} + 2\, \mathcal{Z}_{\ydiagram{3,1,1}} + \mathcal{Z}_{\ydiagram{2,2,1}} + \mathcal{Z}_{\ydiagram{1,1,1,1,1}} \ , \qquad \text{etc.}
\end{aligned} \tag{5.9}$$

As anticipated, only a subset of YT's, those compatible with cyclicity of the trace, enters in (5.9). In particular, HS multiplets associated to the tableaux $\ydiagram{1,1}$, $\ydiagram{2,1}$, two out of the three of type $\ydiagram{2,1,1}$, and so on, are projected out. The content of the various components in (5.9) can be derived from

$$\mathcal{Z}_{\ydiagram{2}} = \frac{1}{2!}\left[\mathcal{Z}_{\ydiagram{1}}(t)^2 + \mathcal{Z}_{\ydiagram{1}}(t^2)\right] \tag{5.10}$$

$$\mathcal{Z}_{\ydiagram{3}} = \frac{1}{3!}\left[\mathcal{Z}_{\ydiagram{1}}(t)^3 + 3\, \mathcal{Z}_{\ydiagram{1}}(t^2)\mathcal{Z}_{\ydiagram{1}}(t) + 2\, \mathcal{Z}_{\ydiagram{1}}(t^3)\right] \tag{5.11}$$

$$\mathcal{Z}_{\ydiagram{1,1,1}} = \frac{1}{3!}\left[\mathcal{Z}_{\ydiagram{1}}(t)^3 - 3\, \mathcal{Z}_{\ydiagram{1}}(t^2)\mathcal{Z}_{\ydiagram{1}}(t) + 2\, \mathcal{Z}_{\ydiagram{1}}(t^3)\right] \tag{5.12}$$

$$\mathcal{Z}_{\ydiagram{4}} = \frac{1}{4!}\left[\mathcal{Z}_{\ydiagram{1}}(t)^4 + 6\, \mathcal{Z}_{\ydiagram{1}}(t^2)\mathcal{Z}_{\ydiagram{1}}(t)^2 + 3\, \mathcal{Z}_{\ydiagram{1}}(t^2)^2 + 8\, \mathcal{Z}_{\ydiagram{1}}(t^3)\mathcal{Z}_{\ydiagram{1}}(t) + 6\, \mathcal{Z}_{\ydiagram{1}}(t^4)\right] \tag{5.13}$$

$$\mathcal{Z}_{\ydiagram{2,2}} = \frac{1}{4!}\left[2\, \mathcal{Z}_{\ydiagram{1}}(t)^4 + 6\, \mathcal{Z}_{\ydiagram{1}}(t^2)^2 - 8\, \mathcal{Z}_{\ydiagram{1}}(t^3)\, \mathcal{Z}_{\ydiagram{1}}(t)\right] \tag{5.14}$$

$$\mathcal{Z}_{\tiny\yng(2,1,1)} = \frac{1}{4!}\left[3\,\mathcal{Z}_{\tiny\yng(1)}(t)^4 - 6\,\mathcal{Z}_{\tiny\yng(1)}(t^2)\mathcal{Z}_{\tiny\yng(1)}(t)^2 - 3\,\mathcal{Z}_{\tiny\yng(1)}(t^2)^2 + 6\,\mathcal{Z}_{\tiny\yng(1)}(t^4)\right] \ . \tag{5.15}$$

that can be explicitly verified with the use of (5.8).

Under the superconformal group $\mathfrak{psu}(2,2|4)$, the HS multiplet $\mathcal{Z}_{YT}$, associated to a given Young tableau YT with L boxes, decomposes into an infinite sum of multiplets. The HWS's can be found by computing $\mathcal{Z}_{YT}$ and eliminating the superconformal descendants by passing $\mathcal{Z}_{YT}$ through a sort of Eratosthenes (super) sieve [1]. In the $\mathfrak{psu}(2,2|4)$ notation $\mathcal{V}^{\Delta,B}_{[j,\bar{\jmath}][q_1,p,q_2]}$ of the Appendix one finds for $L = 2, 3$

$$\begin{aligned}
\mathcal{Z}_{\tiny\yng(2)} &= \sum_{n=0}^{\infty} \mathcal{V}^{2n,0}_{[-1+n^*,-1+n^*][0,0,0]} \ , \\
\mathcal{Z}_{\tiny\yng(3)} &= \sum_{n=0}^{\infty} c_n \left[\mathcal{V}^{1+n,0}_{[-1+\frac{1}{2}n^*,-1+\frac{1}{2}n^*][0,1,0]} + \big(\mathcal{V}^{\frac{11}{2}+n,\frac{1}{2}}_{[\frac{3}{2}+\frac{1}{2}n^*,1+\frac{1}{2}n^*][0,0,1]} + \text{h.c.}\big)\right] \\
&\quad + \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} c_n \left[\mathcal{V}^{4+4m+n,1}_{[1+2m+\frac{1}{2}n^*,\frac{1}{2}n][0,0,0]} + \mathcal{V}^{9+4m+n,1}_{[\frac{7}{2}+2m+\frac{1}{2}n^*,\frac{3}{2}+\frac{1}{2}n][0,0,0]} + \text{h.c.}\right] \ , \\
\mathcal{Z}_{\tiny\yng(1,1,1)} &= \sum_{n=0}^{\infty} c_n \left[\mathcal{V}^{4+n,0}_{[\frac{1}{2}+\frac{1}{2}n^*,\frac{1}{2}+\frac{1}{2}n^*][0,1,0]} + \big(\mathcal{V}^{\frac{5}{2}+n,\frac{1}{2}}_{[\frac{1}{2}n^*,-\frac{1}{2}+\frac{1}{2}n^*][0,0,1]} + \text{h.c}\big)\right] \\
&\quad + \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} c_n \left[\mathcal{V}^{6+4m+n,1}_{[2+2m+\frac{1}{2}n^*,\frac{1}{2}n][0,0,0]} + \mathcal{V}^{7+4m+n,1}_{[\frac{5}{2}+2m+\frac{1}{2}n^*,\frac{3}{2}+\frac{1}{2}n][0,0,0]} + \text{h.c.}\right] \ .
\end{aligned} \tag{5.16}$$

The multiplicities $c_n \equiv 1 + [n/6] - \delta_{n,1 \bmod 6}$ with $[m]$ the integral part of m, of $\mathfrak{psu}(2,2|4)$ multiplets inside $\mathfrak{hs}(2,2|4)$ count the number of ways one can distribute HS descendants among the boxes in the tableaux.

In addition to the $\frac{1}{2}$-BPS multiplet with $n = 0$, the symmetric doubleton $\mathcal{Z}_{\tiny\yng(2)}$, corresponding to the quadratic Casimir δ_{ab}, contains the semishort multiplets of conserved HS currents $\mathcal{V}_{2n}$. The antisymmetric doubleton $\mathcal{Z}_{\tiny\yng(1,1)}$ is ruled out by cyclicity of the trace, cf. (5.9). The 'symmetric tripleton' $\mathcal{Z}_{\tiny\yng(3)}$, corresponding to the cubic Casimir d_{abc}, contains the first KK recurrences of twist 2 semishort multiplets, the semishort-semishort series $\mathcal{V}_{\pm 1,n}$ starting with fermionic primaries and long-semishort multiplets. The antisymmetric tripleton $\mathcal{Z}_{\tiny\yng(1,1,1)}$, corresponding to the structure constants f_{abc}, on the other hand contains the Goldstone multiplets that merge with twist 2 multiplets to form long multiplets when the HS symmetry is broken. In addition, fermionic semishort-semishort multiplets and long-semishort multiplets also appear.

6 Glimpses of *La Grande Bouffe*

The problem of formulating the dynamics of HS fields dates back to Dirac, Wigner, Fierz and Pauli. In the massless bosonic case, Fronsdal has been able to write down linearized field equations for totally symmetric tensors $\varphi^{(\mu_1 \dots \mu_s)}$ that in $D = 4$ arise from the quadratic action [77, 73, 78]

$$
\begin{aligned}
S_2^{(s)} \quad & = \frac{1}{2}(-)^s \int d^4x \, \{\partial_\nu \varphi_{\mu_1 \dots \mu_s} \partial^\nu \varphi^{\mu_1 \dots \mu_s} \\
& - \frac{s(s-1)}{2} \partial_\nu \varphi^\lambda{}_{\lambda \mu_3 \dots \mu_s} \partial^\nu \varphi_\rho^{\rho \mu_3 \dots \mu_s} + s(s-1) \partial_\nu \varphi^\lambda{}_{\lambda \mu_3 \dots \mu_s} \partial_\rho \varphi^{\nu \rho \mu_3 \dots \mu_s} \\
& - s \partial_\nu \varphi^\nu{}_{\mu_2 \dots \mu_s} \partial_\rho \varphi^{\rho \mu_2 \dots \mu_s} - \frac{s(s-1)(s-2)}{4} \partial_\nu \varphi^{\nu \rho}{}_{\rho \mu_2 \dots \mu_s} \partial_\lambda \varphi_\sigma{}^{\lambda \sigma \mu_2 \dots \mu_s} \}
\end{aligned}
\tag{6.1}
$$

upon imposing 'doubly tracelessness' $\eta^{\mu_1 \mu_2} \eta^{\mu_3 \mu_4} \varphi_{\mu_1 \dots \mu_s} = 0$. HS gauge invariance correspond to transformations

$$\delta \varphi_{\mu_1 \dots \mu_s} = \partial_{(\mu_1} \epsilon_{\mu_2 \dots \mu_s)}$$

with traceless paramaters $\eta^{\mu_1 \mu_2} \epsilon_{\mu_1 \dots \mu_{s-1}} = 0$. Fang and Frosdal have then extended the analysis to fermions [79], while Singh and Hagen formulated similar equations for massive fields with the help of auxiliary fields, that reduce to Frondal's or Fang-Fronsdal's in the massless limit upon removing certain auxiliary fields [80, 81]. String theory in flat spacetime can be considered as a theory of an infinite number of HS gauge fields of various rank and (mixed) symmetry in a broken phase. At high energies these symmetries should be restored resulting in a new largely unexplored phase.

Upon coupling to (external) gravity, the presence of the Weyl tensor in the variation of the action for $s > 2$, resulting from the Riemann tensor in the commutator of two covariant derivatives, spoils HS gauge invariance even at the linearized level except for spin $s \leq 2$, where at most the Ricci tensor appears. Problems with interactions for HS gauge fields in flat spacetime are to be expected since the Coleman - Mandula theorem and its generalization by Haag - Lopusanski - Sohnius lead to a trivial S-matrix whenever the (super)Poincarè group is extended by additional spacetime generators such as HS symmetry currents. Moreover closure of the HS algebra requires an infinite tower of symmetries as soon as HS fields with $s > 2$ enter the game. A completely new approach to the interactions, if any, is to be expected in order to deal with an infinite number of HS fields and arbitrarily high derivatives.

According to Fradkin and Vasiliev, the situation improves significantly when the starting point is taken to be a maximally symmetric AdS space[7] with non-vanishing cosmological constant $\Lambda = -(D-2)(D-1)/R^2$ rather than flat spacetime. One can then

[7] Results for dS space formally obtain by analytic continuation.

use the HS analogue of the MacDowell, Mansouri, Stelle, West (MDMSW) $SO(d,2)$ formulation of gravity in order to keep HS gauge symmetry manifest and compactly organize the resulting higher derivative interactions and the associated non-locality. Misha Vasiliev [73, 82–84, 78, 74] has been able to pursue this program till the very end, *i.e.* at the fully non-linear level, for massless bosons in $D = 4$.

The AdS/CFT correspondence at the HS enhancement point seems exactly what the doctor ordered. At generic radius R, superstring theory describes HS fields in a broken phase. At some critical radius, Vasiliev's equations govern the dynamics of the exactly massless phase. Here Λ plays a double role. On the one hand it suppresses higher derivative interactions, very much like the string scale $M_s = 1/\sqrt{\alpha'}$ does in string theory. On the other hand it allows one to define a generalized $SO(d,2)$ curvature (the bulk is $D = d+1$ dimensional) that vanishes exactly for AdS.

In MDMSW formulation, one treats gravity with cosmological constant in $D = d+1$ dimensions as an $SO(d,2)$ gauge theory with a 'compensator' $V^A(x)$ ($A = 0, 1, ...d-1, d, d+1$) such that $\eta_{AB}V^AV^B = -R^2$ with $\eta_{AB} = (-, +, ..., +|+, -)$. To this end, one extends the familiar frame and connection one-forms

$$e^a(x) = dx^m e^a_m(x) \quad , \quad \omega^{ab}(x) = dx^m \omega^{ab}_m(x)$$

with $m, n = 0, 1, ...d$ to

$$E^A = dV^A + \omega^A{}_B V^B = DV^A \ ,$$

so that $E_A V^A = 0$, and

$$\omega^{AB} = \omega^{AB}_L - \Lambda(E^AV^B - E^BV^A)$$

where ω^{AB}_L is the generalized Lorentz $SO(d,1)$ connection, such that $D_L V^A = dV^A + \omega^A_{L\,B}V^B = 0$. In the ' unitary gauge', $V^A = R\delta^A_{d+1}$, $\omega^{ab}_L = \omega^{ab}$ and $e^a = \omega^{aA}V_A$. The generalized curvature two-forms

$$R^A{}_B = d\omega^A{}_B + \omega^A{}_C \wedge \omega^C{}_B$$

contain a longitudinal 'torsion' part $R^A = R^A{}_B V^B = DE^A$ and a transverse 'Lorentz' $SO(d,1)$ part. For (A)dS

$$R^A{}_B = 0$$

with $rk(E^A_m) = d+1$ in order for the existence of a non-degenerate metric tensor $g_{mn} = \eta_{AB}E^A_m E^B_n$. The MDMSW action in $D = d+1$ dimensions is given by

$$S = -\frac{1}{4\kappa\sqrt{\Lambda}} \int_{M_{d+1}} \epsilon_{A_0A_1...A_{d+1}} R^{A_0A_1} \wedge R^{A_2A_3} \wedge E^{A_4} \wedge ... \wedge E^{A_d} V^{A_{d+1}}$$

S is manifestly invariant under diffeomorphisms and, thanks to the compensator V^A, under $SO(d,2)$ gauge transformations

$$\delta\omega^{AB} = D\varepsilon^{AB} \quad , \quad \delta V^A = -\varepsilon^{AB}V_B$$

Fixing the $SO(d,2)$ gauge requires a compensating diffeomorphism so that

$$\delta' V^A = 0 = \delta_\varepsilon V^A + \delta_\xi V^A = \xi^m E^A_m - \varepsilon^{AB}V_B$$

The most symmetric solutions of the field equations are 'flat connections' ω_0^{AB} and correspond to AdS. Global symmetries very much like Killing vectors satisfy $D_0\varepsilon^{AB} = 0$, integrability follows from the zero-curvature condition $R_0{}^A{}_B = 0$.

Generalization to massless HS gauge fields [78] is conveniently described by a set of one-forms $\omega^{a_1...a_{s-1},b_1...b_t}$ à la De Wit and Friedman [85] with $t \leq s-1$ and $a_i, b_i = 0, ..., d$, that represent two-row Young tableaux of $SO(d,1)$ with $s-1$ and t boxes respectively. Dynamical fields are frame type $\varphi^{a_1...a_s} = e^{n(a_s}\omega_n^{a_1...a_{s-1})}$ totally symmetric and (doubly) traceless tensors. Higher t generalized connections are auxiliary and expressible in terms of order t derivatives of $\varphi^{a_1,...a_s}$. In the $SO(d,2)$ invariant formulation, the generalized connections are $\omega^{A_1,...A_{s-1},B_1,...,B_t}$. One can define the linearized HS curvature two-forms

$$R_1^{A_1...A_{s-1},B_1...B_{s-1}} = D_0\omega^{A_1...A_{s-1},B_1...B_{s-1}}$$

where D_0 is the background $SO(d,2)$ covariant derivative with flat connection ω_0^{AB}. The $SO(d,2)$ covariant form of the (quadratic) action for spin s is

$$S_2^{(s)} = \frac{1}{2}\sum_{p=0}^{s-2} a(s,p)\epsilon_{A_0A_1...A_{d+1}} \int_{M_{d+1}} E^{A_4} \wedge ... \wedge E^{A_d} V^{A_{d+1}} V_{C_1}...V_{C_{2(s-2-p)}} \wedge \tag{6.2}$$

$$R^{A_0}{}_{B_1...B_{s-2}}{}^{,A_1C_1...C_{s-2-p}}{}_{D_1...D_p} \wedge R^{A_2B_1...B_{s-2},A_3C_{s-p-1}...C_{2(s-2-p)}D_1...D_p} \tag{6.3}$$

where $a(s,p) = b(s)(-\Lambda)^{-(s-p-1)}[d-5+2(s-p-2)]!!(s-p-1)/(s-p-2)!$ and $b(s)$ is fixed by the so-called 'extra fields decoupling condition', that prevents the propagation of the auxiliary lower spin fields with $t < s$ [78]. Linearized HS gauge invariance under

$$\delta\omega^{A_1...A_{s-1},B_1...B_{s-1}} = D_0\varepsilon^{A_1...A_{s-1},B_1...B_{s-1}}$$

is a consequence of the zero-curvature condition $R_0{}^A{}_B = 0$. In higher dimensions *e.g.* $D = 5$ mixed symmetry tensors and spinors may appear and in fact do so as a consequence e.g. of supersymmetry.

Minimal (bosonic) HS symmetry can be defined as the symmetry of a massless scalar field theory living on the $d(= D-1)$ dimensional boundary of AdS [78]. In the holographic

perspective, global symmetries on the boundary correspond to (global remnants of) local symmetries in the bulk. HS symmetry should correspond to the global symmetry of the maximally symmetric background. In the AdS vacuum, HS symmetry is generically broken to the (super)conformal symmetry, except possibly for the point of enhanced HS symmetry, that should correspond to some small curvature radius.

Bosonic HS symmetry algebras admit generators $T_{A_1...A_s,B_1...B_s}$ in the two-row Young tableaux of $SO(d,2)$ with $s=t$. Generalized commutation relations with the $SO(d,2) \subset HS(d,2)$ generators $T_{A,B}$ take the obvious form

$$[T^C{}_D, T_{A_1...A_n,B_1...B_n}] = \delta^C{}_{A_1} T_{D...A_n,B_1...B_n} + ...$$

For later purposes it is convenient to introduce two sets of oscillators Y_i^A with $i = 1,2$ that satisfy

$$[Y_i^A, Y_j^B]_* = \eta^{AB}\epsilon_{ij}$$

where the Moyal-Weyl $*$-product is defined by

$$(f * g)(Y) = \int \frac{dTdS}{\pi^{2(d+2)}} f(Y+S)g(Y+T)\exp(-2S\cdot T)$$

The associative Weyl algebra $\mathcal{A}_{d+2}$ of polynomials

$$P_n(Y) = \sum_{\{A_p,B_q\}} \varphi_{A_1..A_m,B_1...B_n} Y_1^{A_1}...Y_1^{A_m} Y_2^{B_1}...Y_2^{B_n}$$

contains $so(n,m) \oplus sp(2)$. Moreover, although the subalgebra $\mathcal{A}_{d+2}$ of $sp(2)$ singlets is not simple one can mod out the ideal $\mathcal{I}$ generated by elements of the form $a^{ij} * t_{ij} = t_{ij} * a^{ij}$. Only traceless two-row Yang tableaux appear in the expansion. The Lie algebra with the commutator in $\mathcal{S}/\mathcal{I}$ has a real form denoted by $hu(1/sp(2)[n,m])$, which is the algebra of HS symmetry. In $D=4$ HS gauge fields can be compactly assembled into a connection 'master field' or simply 'master connection'

$$\omega(Y|x) = \sum_{n=0}^{\infty} \varphi_{A_1..A_n,B_1...B_n} Y_1^{A_1}...Y_1^{A_n} Y_2^{B_1}...Y_2^{B_n}$$

Gauge transformations are compactly encoded in

$$\delta\omega(Y|x) = D\varepsilon(Y|x) \equiv d\varepsilon(Y|x) + [\omega(Y|x), \varepsilon(Y|x)]_*$$

The curvature 'master field' is defined as usual

$$R(Y|x) = d\omega(Y|x) + \omega(Y|x) \wedge *\omega(Y|x)$$

The HS symmetry algebra is infinite dimensional and contains $so[d,2] \oplus u(1)$. HS symmetry transformations satisfy the composition rule

$$[\varepsilon(Y|x)_{s_1}, \varepsilon(Y|x)_{s_2}]_* = \sum_{t=|s_1-s_2|+1}^{s_1+s_2-2} \varepsilon(Y|x)_t$$

As soon as $s > 2$ the algebra becomes infinite dimensional. The Moyal-Weyl structure naturally leads to non-commutativity that in turn can be related to the non-locality resulting from interactions involving higher and higher derivatives. To some extent a massless HS gauge theory is in between a field theory, with spin ranging from 0 to 2, and string theory, where spin is not bounded. Strings in AdS are the natural arena where to put this analogy at work.

In the MDMSW description [78] one can treat the Weyl tensor $C^{AC,BD}$ as an independent field and deform the field equations according to

$$R^{A,B}|_{oms} = E_C \wedge E_D C^{AC,BD}$$

where *oms* stands for 'on mass-shell'. Similarly 'unfolded' HS dynamics can be defined in terms of generalized Weyl tensors $C^{A_1..A_s,B_1...B_s}$ (two-row Young tableaux), that satisfy the first on-mass-shell (oms) theorem

$$R^{A_1..A_{s-1},B_1...B_{s-1}}|_{oms} = E_{A_s} \wedge E_{B_s}\ C^{A_1..A_s,B_1...B_s}$$

Linearizing around a maximally symmetric background yields

$$R_1^{A_1...A_{s-1},B_1...B_{s-1}} = E_{0A_s} \wedge E_{0B_s} C^{A_1...A_s,B_1...B_s}$$

The full set of compatibility conditions on the basis $C^{A_1...A_u,B_1...B_s}$ of $u-s$ derivatives of $C^{A_1...A_s,B_1...B_s}$ is subsumed by

$$\widetilde{D}_0 C^{A_1...A_u,B_1...B_s} = 0 \quad , \quad u \geq s$$

where

$$\widetilde{D}_0 = D_{0L} - \Lambda E_0^A V^B (2Y_A^{i\perp} Y_{Bi}^{\parallel} + \frac{1}{2}\varepsilon^{ij}\partial_{iA}^{\perp}\partial_{jB}^{\parallel})$$

is the background covariant derivative in the so-called '*twisted* adjoint representation', where $\partial_{iA} = \partial/\partial Y^{iA}$ and $X^{A\parallel} = V^A V_B X^B$ and $X^{A\perp} = X^A - X^{A\parallel}$.

At the linearized level, the 'central on-mass-shell theorem' implies

$$R_1(Y^{\parallel}, Y^{\perp})|x) = \frac{1}{2} E_0^A \wedge E_0^B \varepsilon^{ij} \partial_{iA}\partial_{jB} C(0, Y^{\perp}|x)$$

and

$$\widetilde{D}_0 C(Y|x) = 0$$

where $\widetilde{D}_0 C \equiv dC + \omega_0 * C - C * \widetilde{\omega}_0$ and $R_1 \equiv d\omega + \omega_0 \wedge *\omega + \omega \wedge *\widetilde{\omega}_0$

In Vasiliev's formulation [78] of non-linear HS dynamics in $D = 4$, one doubles the number of oscillators, adding Z_i^A with $[Z_i^A, Z_j^B]_* = -\eta^{AB}\epsilon_{ij}$, and works with three 'master fields': the connection one-form $W(Z,Y|x) = dx^n W_n(Z,Y|x)$ such that $W(0,Y|x) = \omega(Y|x)$, the 0-form (scalar) $B(Z,Y|x)$ such that $B(0,Y|x) = C(Y|x)$, and an 'auxiliary' one-form $S(Z,Y|x) = dZ_i^A S_A^i(Z,Y|x)$. One then generalizes the Moyal-Weyl star product

$$(f * g)(Z,Y) = \int \frac{dTdS}{\pi^{2(d+2}} f(Z+S, Y+S) g(Z-T, Y+T) \exp(-2S \cdot T)$$

and defines the 'inner Klein operator'

$$\mathcal{K} = \exp(-2z_i y^i)$$

with $x_i = V_A X_i^A / \sqrt{V \cdot V}$ such that

$$\mathcal{K} * f = \widetilde{f} * \mathcal{K} \quad , \quad \mathcal{K} * \mathcal{K} = 1$$

where $\widetilde{f}(Z,Y) = f(\widetilde{Z}, \widetilde{Y})$ with $\widetilde{X}^A = X^{A\perp} - X^{A\parallel}$. The full non-linear system of Vasiliev's HS equations then reads

$$dW + W * W = 0 \quad , \quad dS + W * S + S * W = 0 \quad , \quad dB + W * B - B * \widetilde{W} = 0$$

and

$$S * S = -\frac{1}{2}(dZ_i^A dZ_A^i + \frac{4}{\Lambda} dz_i dz^i B * \mathcal{K}) \quad , \quad S * B = B * \widetilde{S}$$

In terms of $\mathcal{W} = d + W + S$ one can combine both sets of equations into

$$\mathcal{W} * \mathcal{W} = -\frac{1}{2}(dZ_i^A dZ_A^i + \frac{4}{\Lambda} dz_i dz^i B * \mathcal{K}) \quad , \quad \mathcal{W} * B = B * \widetilde{\mathcal{W}}$$

Formal consistency follows from associativity, while gauge invariance under

$$\delta \mathcal{W} = [\varepsilon, \mathcal{W}]_* \quad , \quad \delta B = \varepsilon * B - B * \widetilde{\varepsilon}$$

is manifest. It is remarkable that the fully non-linear dynamics is presented in the form of a zero-curvature condition that leads to an integrable Cartan system.

The linearized HS field equations resulting from the positions

$$W = W_0 + W_1 \quad , \quad S = S_0 + S_1 \quad , \quad B = B_0 + B_1$$

with

$$W_0 = \frac{1}{2}\omega_0^{AB} Y_A^i Y_{iB} \quad , \quad B_0 = 0 \quad S_0 = dZ_A^i Z_{iA}$$

where ω_0 has zero $SO(d,2)$ curvature so as to describe AdS_{d+1}, are equivalent to Fronsdal's component HS field equations in an AdS background.

Vasiliev's equations [78] describe HS gauge fields in $D = 4$ where only totally symmetric tensors are relevant. On the one hand one would like to extend his analysis to the case of $HS(2,2|4)$ relevant to $\mathcal{N} = 4$ SYM in $d = 4$, $D = 5$ bulk. Sezgin and Sundell [70, 71, 45] have been able to write down field equations for the 'massless' $HS(2,2|4)$ doubleton ($L = 2$). As in Vasiliev's case the field content can be assembled into a master connection A and a master scalar (curvature) Φ. The former transform in the adjoint representation of $hs(2,2|4)$ and contains physical gauge fields with $s \geq 1$ and $B = 0, \pm 1$. The latter transform in the twisted adjoint representation and contributes physical fields with spin $s \leq 1/2$ or $s \geq 1$ but $|B| \geq 3/2$ (self-dual two-form potentials). The field strengths

$$F_A = dA + A \wedge *A \quad , \quad D_A\Phi = d\Phi + A * \Phi - \Phi * \widetilde{A} \tag{6.4}$$

transform covariantly

$$\delta F_A = [F_A, \epsilon]_* \quad , \quad \delta D_A\Phi = D_A\Phi * \widetilde{\epsilon} - \epsilon * D_A\Phi \tag{6.5}$$

under gauge transformations

$$\delta A = d\epsilon + [A, \epsilon]_* \quad , \quad \delta\Phi = \Phi * \widetilde{\epsilon} - \epsilon * \Phi \tag{6.6}$$

The linearized constraints and integrability conditions lead after some tedious algebra to the correct linearized field equations for the 'matter' fields with $s \leq 1/2$, for the HS gauge fields and for the antisymmetric tensors with generalized self-duality.

Possibly because of the presence of these generalized (anti)self dual tensors, an interacting $hs(2,2|4)$ gauge theory has not yet been formulated. For the purpose of describing the breaking of $hs(2,2|4)$ to $psu(2,2|4)$, however, one is rather interested in the coupling of the 'massive' HS multiplets (totally antisymmetric 'tripleton' and window-like 'tetrapleton') containing the Goldstone lower spin modes to the massless doubleton at large N. This problem might turn out to be easier to solve than constructing a fully non-linear massless $hs(2,2|4)$ theory because it should be fixed by linearized HS symmetry and require a little bit more than the knowledge of the linearized field equations. In some sense equations of Vasiliev's type should encode combinatorial interactions which are present even in a free field theory at finite N, where HS symmetry is unbroken, or couplings to multi-particle states[8]. From a holographic perspective the interactions which are responsible for the breaking of HS symmetry are equivalent to making the curvature

[8]Precisely for this reason they are relevant in the $d = 3$ model on the boundary of AdS_4 considered by Klebanov and Polyakov, for a recent review see *e.g.* [86].

radius larger than the string scale. Although truncation to the HS massless multiplet (doubleton) should be consistent at the point of HS enhancement this should no more be the case for generic R in AdS_5.

When interactions are turned on only one out of the infinite tower of conserved current doubleton multiplets in $\mathcal{N}=4$ SYM theory

$$\mathcal{Z}_{\square\!\square} = \sum_{n=0}^{\infty} \mathcal{V}_{2n}, \qquad \mathcal{V}_j := \mathcal{V}^{j,0}_{[-1+\frac{1}{2}j^*,-1+\frac{1}{2}j^*][0,0,0]}. \tag{6.7}$$

is protected against quantum corrections to the scaling dimension: the $\mathcal{N}=4$ supercurrent multiplet $\mathcal{V}_0 = \mathcal{V}^{2,0}_{[0^\dagger,0^\dagger][0^\dagger,2,0^\dagger]}$. The remaining multiplets $\mathcal{V}_{2n}$ acquire anomalous dimensions dual to mass shifts in the bulk which violate the conservation of their HS currents at the quantum level. At one-loop, one has [87, 88, 56]

$$\gamma_{1-\text{loop}}(2n) = \frac{g^2_{\text{YM}}N}{2\pi^2}\, h(2n), \qquad h(j) = \sum_{k=1}^{j} \frac{1}{k}, \tag{6.8}$$

This elegant ('number theoretic') formula gives a clue on how to compute generic anomalous dimensions at first order in perturbation theory relying on symmetry breaking considerations. Naively, one would look for all occurrences of the broken currents $\mathcal{V}_{2n}$ within some operator $\mathcal{O}$. Each occurrence of some broken current should contribute to the anomalous dimension of $\mathcal{O}$ a term proportional to $h(2n)$. Indeed, this is nearly what happens, the one-loop dilatation operator [69] can be written as

$$H = \sum_{s=1}^{L} H_{(s,s+1)} = \sum_{s=1}^{L}\sum_{j=0}^{\infty} 2h(j)\, P^{j}_{(s,s+1)}, \tag{6.9}$$

where $P^j_{(s,s+1)}$ projects the product of fields ('letters') at nearest neighboring sites s and $s+1$ onto $\mathcal{V}_j$. Here, the sum goes over all values of j and not just the even ones. The point is that although bilinear currents $\mathcal{V}_{2n+1}$ corresponding to the broken generators are eliminated after tracing over color indices, they still appear in subdiagrams $\begin{smallmatrix}\square\\ \square\end{smallmatrix}$ inside a bigger trace.

In order to achieve a holographic description of *La Grande Bouffe*, our previous identification of the necessary 'longitudinal' modes in the AdS bulk, turns out to be crucial. Using the Konishi multiplet as a prototype, one expects something like

$$\mathcal{K}_{long} \leftrightarrow \mathcal{K}_{short} + \mathcal{K}_{1/4} + \mathcal{K}_{1/8} + \mathcal{K}^*_{1/8} \tag{6.10}$$

i.e. HS semishort multiplets, such as $\mathcal{K}_{short}$, eat lower spin Goldstone multiplets, such as $\mathcal{K}_{1/8}$, its conjugate $\mathcal{K}^*_{1/8}$ and $\mathcal{K}_{1/4}$. Although massless HS fields with mixed symmetry have

been only recently addressed [89], whenever they are part of the HS doubleton multiplet, supersymmetry should be enough to determine their equations from the more familiar equations for symmetric tensors. In particular for the $\mathcal{N}=4$ Konishi multiplet we know the axial anomaly is part of an on-shell anomaly supermultiplet

$$\bar{D}^A\bar{D}^B\mathcal{K}_{long} = g_{ym}Tr(\mathcal{W}_{EF}[\mathcal{W}^{AE}\mathcal{W}^{BF}]) + \frac{g_{ym}^2}{8\pi^2}D_E D_F Tr(\mathcal{W}^{AE}\mathcal{W}^{BF}) \tag{6.11}$$

The formulation of *La Grande Bouffe* we have in mind is of the Stückelberg type [90–92]. Let us illustrate it for the broken singlet current in the $\mathcal{N}=4$ Konishi multiplet. The bulk Lagrangian describing the holographic Higgs mechanism à la Stückelberg should schematically be of the form

$$L = -\frac{1}{4}F(V)^2 + \frac{1}{2}(\partial\alpha - MV)^2 \tag{6.12}$$

where F_{mn} is the field-strength of the bulk vector field V_n dual to the current J_μ and α is the bulk scalar dual to the 'anomaly' $A = \partial_\mu J^\mu$. Gauge invariance under

$$\delta V_m = \partial_m\vartheta \quad , \quad \delta\alpha = M\vartheta \tag{6.13}$$

is manifest for constant M. For $M=0$, V and α decouple. For $M\neq 0$, V eats α and becomes massive. In practice M should depend on the dilaton and the other massless scalars. Since we want to preserve superconformal invariance, M can at most acquire a constant vev and the above analysis for a vector current seems robust. Generalization to higher spins and AdS covariantization should not pose any fundamental problem. We will use string theory in flat space as a guidance in order to write down gauge invariant field equations that allow for symmetry breaking à la Stückelberg.

Let us start from the field equations for a massive HS field of spin s corresponding to a totally symmetric traceless tensor, *i.e.* $\phi_{(s)}$ with $\hat{\phi}_{(s-2)} = 0$. Suppressing indices as in [93, 94], one has

$$\partial^2_{(0)}\phi_{(s)} + M^2\phi_{(s)} = 0 \quad , \quad \partial_{(-1)}\phi_{(s)} = 0 \quad . \tag{6.14}$$

For spin s, the minimal set of auxiliary fields identified by Singh and Hagen [80, 81] consists in $\phi_{(s-t)} \approx \partial^t_{(-1)}\phi_{(s)}$ with $t = 2, ...s$. The Singh - Hagen Lagrangian reads

$$\begin{aligned}(-)^s L_{(s)} &= \frac{1}{2}(\partial_{(1)}\phi_{(s)})^2 - \frac{s}{2}(\partial_{(-1)}\phi_{(s)})^2 - \frac{1}{2}M^2\phi^2_{(s)} \\ &+C_s\{a_2M^2\phi^2_{(s-2)} - \frac{1}{2}(\partial_{(1)}\phi_{(s-2)})^2 + \phi_{(s-2)}\partial^2_{(-1)}\phi_{(s)} + \frac{b_2}{2}(\partial_{(-1)}\phi_{(s-2)})^2 \\ &-\sum_{t=3}^{s}\prod_{r=2}^{t-1}c_r[\frac{1}{2}(\partial_{(1)}\phi_{(s-t)})^2 - a_tM^2\phi^2_{(s-t)} - \frac{b_t}{2}(\partial_{(-1)}\phi_{(s-t)})^2 - M\phi_{(s-t)}\partial_{(-1)}\phi_{(s-t+1)}]\}\end{aligned} \tag{6.15}$$

where $c_t = -(t-1)(s-t)^2(s-t+2)(2s-t+2)/[2(s-t+1)(2s-2t+1)(2s-2t+3)]$, $C_s = s(s-1)^2/(2s-1)$, $a_t = t(2s-t+1)(s-t+2)/[2(2s-2t+3)(s-t+1)]$, and $b_t = -(s-t)^2/(2s-2t+3)$.

In the massless limit, $M \to 0$, HS equations enjoy gauge invariance. The spin 1 case is very well known. For spin 2, the relevant gauge transformations read $\delta\phi_{(2)} = \partial_{(1)}\epsilon_{(1)} - \frac{\eta_{(2)}}{2}\partial_{(-1)}\epsilon_{(1)}$, so as to preserve tracelessness, and $\delta\phi_{(0)} = -\frac{3}{2}\partial_{(-1)}\epsilon_{(1)}$. It is convenient to introduce a new traceful field $h_{(2)} = \phi_{(2)} - \eta_{(2)}\phi_{(0)}/3$, for which $\delta h_{(2)} = \partial_{(1)}\epsilon_{(1)}$. The 'new' field equations are nothing but linearized Einstein equations

$$\partial^2_{(0)}h_{(2)} - 2\partial_{(1)}\partial_{(-1)}h_{(2)} + \partial^2_{(1)}\hat{h}_{(0)} = 0 \quad . \tag{6.16}$$

For massless spin s, Fronsdal's field equations

$$\partial^2_{(0)}h_{(s)} - s\partial_{(1)}(\partial_{(-1)}h_{(s)}) + \frac{s(s-1)}{2}\partial^2_{(1)}\hat{h}_{(s-2)} = 0 \tag{6.17}$$

with $\hat{\hat{h}}_{(s-4)} = 0$ are invariant under $\delta h_{(s)} = \partial_{(1)}\epsilon_{(s-1)}$ with $\hat{\epsilon}_{(s-3)} = 0$. For $M = 0$, $\phi_{(s)}$ and $\phi_{(s-2)}$ in Singh - Hagen description decouple from the rest. Introducing $h_{(s)} = \phi_{(s)} - \eta_{(2)}\hat{\phi}_{(s-2)}/(2s-1)$ one gets

$$\begin{aligned}(-)^s L_{(s)} &= \frac{1}{2}(\partial_{(1)}h_{(s)})^2 - \frac{s}{2}(\partial_{(-1)}h_{(s)})^2 \\ -\frac{s(s-1)}{2}(\partial_{(1)}\hat{h}_{(s-2)})^2 &- \frac{s(s-1)(s-2)}{8}(\partial_{(-1)}\hat{h}_{(s-2)})^2 - \frac{s(s-1)}{2}\hat{h}_{(s-2)}\partial^2_{(-1)}h_{(s)}\end{aligned} \tag{6.18}$$

that coincides with (6) after suppressing indices.

The set of Stückelberg fields that participate in the spontaneous breaking of HS symmetry can be elegantly derived by formal KK reduction of the (quadratic) massless HS lagrangian from $D+1$ dimensions. The *a priori* complex modes

$$h_{(s)}(x,y) = \sum_{t,M}\psi_{(s-t),M}(x)\exp(iMy) \tag{6.19}$$

generate a bunch of lower spin modes that are needed in order to reproduce the correct number of d.o.f.'s $\nu_{M\neq 0}(d,s)$. It is easy to check that $\nu_{M\neq 0}(d,s) = \nu_{M=0}(d,s) + \nu_{M\neq 0}(d,s-1)$. By iteration, one eventually gets $\nu_{M\neq 0}(d,s) = \sum_{t=0}^{s}\nu_{M=0}(d,t)$. After reduction, *i.e.* integration over y, one can assume that all the ψ's are real for simplicity. More explicitly, from a massless spin s field $\Phi_{(s)}$ in $D+1$ dimensions one gets 'massless' fields $\phi_{(s-t)}$ in D dimensions

$$\Phi_{(0)} \quad \to \quad \{\phi_{(s-t)},\, t = 0, ...s\} \tag{6.20}$$

that altogether provide the d.o.f.'s of a massive spin s field when the double tracelessness conditions

$$\hat{\hat{\Phi}}_{(s)} = 0 \quad \rightarrow \quad \{\hat{\hat{\phi}}_{(s-t-4)} + 2\hat{\phi}_{(s-t-4)} + \phi_{(s-t)} = 0,\, t = 0, ...s-4\} \tag{6.21}$$

are taken into account. The resulting field equations

$$\partial^2_{(0)}\Phi_{(s)} - s?\partial_{(1)}(\partial_{(-1)}\Phi_{(s)}) + \partial_{(1)}(\partial_{(1)}\hat{\Phi}_{(s-2)}) = 0 \quad \rightarrow \tag{6.22}$$

$$\begin{aligned} &\partial^2_{(0)}\Phi_{(s-t)} - \frac{(s-1)(s-2)}{2}M^2\Phi_{(s-t)} + \frac{s(s-1)}{2}M^2\hat{\Phi}_{(s-t+2-2)} \\ &-(s-t)?\partial_{(1)}(\partial_{(-1)}\Phi_{(s-t)}) + \partial_{(1)}(\partial_{(1)}\hat{\Phi}_{(s-t-2)}) + \partial_{(1)}(\partial_{(1)}\Phi_{(s-t-2)}) \\ &-tM\partial_{(-1)}\Phi_{(s-t+1)} + tM\partial_{(-1)}\hat{\Phi}_{(s-t+1)} + (1-t)M(\partial_{(1)}\Phi_{(s-t-1)}) = 0 \end{aligned} \tag{6.23}$$

are invariant by construction under gauge transformations

$$\delta\Phi_{(s)} = \partial_{(1)}\mathcal{E}_{(s-1)} \quad \rightarrow \quad \{\delta\phi_{(s-t)} = \partial_{(1)}\epsilon_{(s-t-1)} + tM\epsilon_{(s-t)},\, t = 0, ...s\} \tag{6.24}$$

with

$$\hat{\mathcal{E}}_{(s-3)} = 0 \quad \rightarrow \quad \{\hat{\epsilon}_{(s-t-3)} + \epsilon_{(s-t-3)},\, t = 0, ...s-1\} \tag{6.25}$$

that expose the role of the lower spin fields in the Higgsing of the HS symmetry.

In string theory the situation is slightly different [93, 94]. Given the difficulties with the quantization of the type IIB in AdS_5, let us take the open bosonic string in flat space as a toy model. States $|\Phi\rangle$ of zero ghost number can be obtained by acting with 'positive' frequency modes on the tachyonic groundstate $|\Omega\rangle = c_{+1}|0\rangle_{SL(2)}$, annihilated by the 'negative' frequency modes of the bosonic coordinates ($\alpha^\mu_n|\Omega\rangle = 0$ with $n > 0$) and the ghosts ($c_n|\Omega\rangle = 0$ for $n > 0$, $b_n|\Omega\rangle = 0$ for $n \geq 0$). Expanding in levels, $|\Phi\rangle = \sum_\ell |\Phi\rangle_\ell$, yields

$$|\Phi\rangle_{\ell=0} = T|\Omega\rangle \tag{6.26}$$

$$|\Phi\rangle_{\ell=1} = [A_\mu\alpha^\mu_{-1} + \rho c_0 b_{-1}]|\Omega\rangle \tag{6.27}$$

$$|\Phi\rangle_{\ell=2} = [H_{\mu\nu}\alpha^\mu_{-1}\alpha^\nu_{-1} + B_\mu\alpha^\mu_{-2} + \psi c_{-1}b_{-1} + \chi_\mu\alpha^\mu_{-1}c_0b_{-1} + \eta c_0 b_{-2}]|\Omega\rangle \tag{6.28}$$

$$\begin{aligned} |\Phi\rangle_{\ell=3} = [&S_{\mu\nu\rho}\alpha^\mu_{-1}\alpha^\nu_{-1}\alpha^\rho_{-1} + U_{\mu\nu}\alpha^\mu_{-2}\alpha^\nu_{-1} + V_\mu\alpha^\mu_{-3} \\ &+\varphi_{\mu\nu}\alpha^\mu_{-1}\alpha^\nu_{-1}c_0b_{-1} + \omega_\mu\alpha^\mu_{-2}c_0b_{-1} + \sigma_\mu\alpha^\mu_{-1}c_0b_{-2} + \tau_\mu\alpha^\mu_{-1}c_{-1}b_{-1} \\ &+\zeta c_0b_{-3} + \vartheta c_{-1}b_{-2} + \gamma c_{-2}b_{-1}]|\Omega\rangle \end{aligned} \tag{6.29}$$

for the first few levels. Field equations $Q|\Phi\rangle = 0$, with $Q = c_0(L_0 - 1) + b_0 M + Q'$, are invariant under gauge transformations $\delta|\Phi\rangle = Q|\Lambda\rangle$, with parameter $|\Lambda\rangle = \sum_\ell |\Lambda\rangle_\ell$ of the

form

$$|\Lambda\rangle_{\ell=0} = 0 \tag{6.30}$$

$$|\Lambda\rangle_{\ell=1} = [\lambda b_{-1}]|\Omega\rangle \tag{6.31}$$

$$|\Lambda\rangle_{\ell=2} = [\varepsilon_\mu \alpha^\mu_{-1} b_{-1} + \theta b_{-2}]\alpha^\mu_{-1} \tag{6.32}$$

$$|\Lambda\rangle_{\ell=3} = [\epsilon_{\mu\nu}\alpha^\mu_{-1}\alpha^\nu_{-1}b_{-1} + \kappa_\mu \alpha^\mu_{-1} b_{-2} + \xi_\mu \alpha^\mu_{-2} b_{-1} + \mu b_{-3} + \nu c_0 b_{-1} b_{-2}$$
$$+\beta b_{-1} b_{-2}]|\Omega\rangle \tag{6.33}$$

The last term $|\Lambda\rangle^{red}_{\ell=3} = \beta b_{-1} b_{-2}|\Omega\rangle$ accounts for the reducibility of the gauge transformation of the antisymmetric tensor present at $\ell = 3$. Fields and gauge parameters involving c_0 appear algebraically and can be eliminated [93, 94].

In components, setting $\alpha' = 1/2$, one finds the following equations.

$$\ell = 0: \qquad \partial^2 T - 2T = 0 \qquad \delta T = 0 \tag{6.34}$$

which is the Klein-Gordon equation for the open string tachyon T.

$$\ell = 1: \qquad \partial^2 A_\mu - \partial_\mu \rho = 0 \qquad \partial^\mu A_\mu - \rho = 0 \tag{6.35}$$

$$\delta A_\mu = \partial_\mu \lambda \qquad \delta\rho = \partial^2 \lambda \tag{6.36}$$

which gives the familiar Maxwell equations and gauge transformations for A_μ after eliminating the 'auxiliary' field ρ.

$$\ell = 2: \qquad (\partial^2 + 2)H_{\mu\nu} - (\partial_\mu \chi_\nu + \partial_\nu \chi_\mu) = 0 \tag{6.37}$$

$$(\partial^2 + 2)B_\mu - 2\partial_\mu \eta = 0 \tag{6.38}$$

$$(\partial^2 + 2)\psi + 2\partial^\mu \chi_\mu + 2\eta = 0 \tag{6.39}$$

$$\partial^\mu H_{\mu\nu} + B_\nu - \chi_\nu = 0 \tag{6.40}$$

$$H^\mu{}_\mu + 2\partial^\mu B_\mu + \eta = 0 \tag{6.41}$$

Elimination of the 'auxiliary' fields χ_μ and η yields a triplet of equations for a massive spin 2 field that can be recast in Singh - Hagen form after some field redefinition.

The same situation prevails for the massive spin 3 field at level $\ell = 3$ after elimination of the 'auxiliary' fields $\varphi_{\mu\nu}$, ω_μ, σ_μ, and ζ and some residual field redefinition.

The emergence in the tensionless limit of triplets of massless covariant field equations

$$\partial^2_{(0)}\phi_{(s)} = \partial_{(1)}\chi_{(s-1)} \quad , \quad \partial_{(-1)}\phi_{(s)} = \chi_{(s-1)} + \partial_{(1)}\psi_{(s-2)} \quad , \quad \partial^2_{(0)}\psi_{(s-2)} = \partial_{(-1)}\chi_{(s-1)} \tag{6.42}$$

with unconstrained gauge invariance

$$\delta\phi_{(s)} = \partial_{(1)}\xi_{(s-1)} \quad , \quad \delta\chi_{(s-1)} = \partial^2_{(0)}\xi_{(s-1)} \quad , \quad \delta\psi_{(s-2)} = \partial_{(-1)}\xi_{(s-1)} \tag{6.43}$$

for totally symmetric fields $\phi_{(s)}$, $\chi_{(s-1)}$, $\psi_{(s-2)}$ in the first Regge trajectory has been recently discussed [95] and reviewed in [94].

AdS covariantization for massless HS fields, as recently discussed in [96, 95], can be achieved by keeping simple gauge transformations

$$\delta\phi_{(s)} = \nabla_{(1)}\xi_{(s-1)} \quad , \quad \delta\psi_{(s-2)} = \nabla_{(-1)}\xi_{(s-1)} \tag{6.44}$$

and constraint

$$\chi_{(s-1)} = \nabla_{(-1)}\phi_{(s)} - \nabla_{(+1)}\psi_{(s-2)} \quad . \tag{6.45}$$

The price one has to pay is a more complicated gauge transformation for $\chi_{(s-1)}$

$$\delta\chi_{(s-1)} = \nabla^2_{(0)}\xi_{(s-1)} + \frac{(s-1)(3-s-D)}{L^2}\xi_{(s-1)} + \frac{2}{L^2}g_{(2)}\hat{\xi}_{(s-3)} \tag{6.46}$$

where $g_{(2)}$ is the AdS metric and field equations for $\phi_{(s)}$ and $\psi_{(s-2)}$

$$\nabla^2_{(0)}\phi_{(s)} = \nabla_{(1)}\chi_{(s-1)} + \frac{1}{L^2}\{8g_{(2)}\psi_{(s-2)} - 2g_{(2)}\hat{\phi}_{(s-2)} + [(2-s)(3-s-D) - s]\phi_{(s)}\} \tag{6.47}$$

$$\nabla^2_{(0)}\psi_{(s-2)} = \nabla_{(-1)}\chi_{(s-1)} + \frac{1}{L^2}\{[s(D+s-2)+6]\psi_{(s-2)} - 2g_{(2)}\hat{\psi}_{(s-4)} - 4\hat{\phi}_{(s-2)}\}. \tag{6.48}$$

Many if not all the ingredients for La Grande Bouffe are at hand, it only remains to put them together and cook them up.

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A $\mathcal{N} = 4$ shortening

Here we collect some notation for representations of the $\mathcal{N} = 4$ superconformal algebra $PSU(2,2|4)$ and their shortenings. We denote by $\mathcal{V}^{\Delta,B}_{[j,\bar{\jmath}][q_1,p,q_2]}$ a generic long multiplet of $\mathfrak{psu}(2,2|4)$ with HWS of conformal dimension Δ and hypercharge B in the $\mathcal{R}_{[j,\bar{\jmath}][q_1,p,q_2]}$ representation of $\mathfrak{su}(2) \times \mathfrak{su}(2) \times \mathfrak{su}(4)$. As usual, $[q_1, p, q_2]$ are Dynkin labels of $\mathfrak{su}(4)$ while $[j, \bar{\jmath}]$ denote $\mathfrak{su}(2) \times \mathfrak{su}(2)$ spins. The representation content of $\mathcal{V}^{\Delta,B}_{[j,\bar{\jmath}][q_1,p,q_2]}$ under the bosonic subalgebra $\mathfrak{su}(2) \times \mathfrak{su}(2) \times \mathfrak{su}(4)$ may be found from evaluating the tensor product $\mathcal{V}^{2,0,0}_{[0,0][0,0,0]}\chi^{\Delta-2,B,P}_{[j,\bar{\jmath}][q_1,p,q_2]}$, with the long Konishi multiplet $\mathcal{V}^{2,0,0}_{[0,0][0,0,0]}$, or explicitly by using the Racah-Speiser algorithm as

$$\mathcal{V}^{\Delta,B}_{[j,\bar{\jmath}][q_1,p,q_2]} = \sum_{\epsilon_{A\alpha},\bar{\epsilon}^A_{\dot{\alpha}}\in\{0,1\}} \chi_{[j,\bar{\jmath}][q_1,p,q_2]+\epsilon_{A\alpha}\mathcal{Q}^A_\alpha+\bar{\epsilon}^A_{\dot{\alpha}}\bar{\mathcal{Q}}_{A\dot{\alpha}}} \,, \tag{A.1}$$

where $\chi_{[j,\bar{\jmath}][q_1,p,q_2]}$ are character polynomials and the sum runs over the 2^{16} combinations of the 16 supersymmetry charges $\mathcal{Q}^A{}_\alpha$, $\bar{\mathcal{Q}}_{A\dot{\alpha}}$, $A = 1,\ldots,4, \alpha, \dot{\alpha} = 1, 2$. In oscillator notation[9]

$$\begin{aligned}
\mathcal{Q}^1{}_\alpha &= a^\dagger_\alpha\, c_1 \equiv \frac{a_\alpha}{c_1} = [\pm\tfrac{1}{2}, 0][1, 0, 0] \,, \quad \bar{\mathcal{Q}}_{1\dot{\alpha}} = b^\dagger_{\dot{\alpha}}\, c^\dagger_1 \equiv b_{\dot{\alpha}}\, c_1 = [0, \pm\tfrac{1}{2}][-1, 0, 0] \\
\mathcal{Q}^2{}_\alpha &= a^\dagger_\alpha\, c_2 \equiv \frac{a_\alpha}{c_2} = [\pm\tfrac{1}{2}, 0][-1, 1, 0] \,, \quad \bar{\mathcal{Q}}_{2\dot{\alpha}} = b^\dagger_{\dot{\alpha}}\, c^\dagger_2 \equiv b_{\dot{\alpha}}\, c_2 = [0, \pm\tfrac{1}{2}][1, -1, 0] \,, \\
\mathcal{Q}^3{}_\alpha &= a^\dagger_\alpha\, d^\dagger_1 \equiv a_\alpha\, d_1 = [\pm\tfrac{1}{2}, 0][0, -1, 1] \,, \quad \bar{\mathcal{Q}}_{3\dot{\alpha}} = b^\dagger_{\dot{\alpha}}\, d_1 \equiv \frac{b_{\dot{\alpha}}}{d_1} = [0, \pm\tfrac{1}{2}][0, 1, -1] \,, \\
\mathcal{Q}^4{}_\alpha &= a^\dagger_\alpha\, d^\dagger_2 \equiv a_\alpha\, d_2 = [\pm\tfrac{1}{2}, 0][0, 0, -1] \,, \quad \bar{\mathcal{Q}}_{4\dot{\alpha}} = b^\dagger_{\dot{\alpha}}\, d_2 \equiv \frac{b_{\dot{\alpha}}}{d_2} = [0, \pm\tfrac{1}{2}][0, 0, 1]
\end{aligned} \tag{A.2}$$

where a_α, $b_{\dot{\alpha}}$, respectively c_r, $d_{\dot{r}}$ appear in the decomposition of y^a under $\mathfrak{su}(2) \times \mathfrak{su}(2) \subset \mathfrak{su}(2,2)$ and respectively of θ^A under $\mathfrak{su}(2) \times \mathfrak{su}(2) \subset \mathfrak{su}(4)$. Every $\mathcal{Q}^A{}_\alpha$, $\bar{\mathcal{Q}}_{A\dot{\alpha}}$ raises the conformal dimension by $\frac{1}{2}$, parity is left invariant, and hypercharge B is lowered and raised by $\frac{1}{2}$ by each $\mathcal{Q}^A{}_\alpha$ and $\bar{\mathcal{Q}}_{A\dot{\alpha}}$ respectively. In order to make sense out of (A.1) also for small values of $q_1, p, q_2, j, \bar{\jmath}$, we note that the character polynomials with negative Dynkin

[9]Notice the flip of notations for the conjugate charges with respect to [1] and the unconventional use of oscillators in the denominator to mean annihilation operators.

labels are to be defined according to

$$\begin{aligned}\chi_{[j,\bar{\jmath}][q_1,p,q_2]} &= -\chi_{[j,\bar{\jmath}][-q_1-2,p+q_1+1,q_2]} \;=\; -\,\chi_{[j,\bar{\jmath}][q_1,p+q_2+1,-q_2-2]} \\ &= -\chi_{[j,\bar{\jmath}][q_1+p+1,-p-2,q_2+p+1]} \\ &= -\chi_{[-j-1,\bar{\jmath}][q_1,p,q_2]} \;=\; -\,\chi_{[j,-\bar{\jmath}-1][q_1,p,q_2]}\;. \end{aligned} \tag{A.3}$$

In particular, this implies that $\chi_{[j,\bar{\jmath}][q_1,p,q_2]}$ is identically zero whenever any of the weights q_1, p, q_2 takes the value -1 or one of the spins j, $\bar{\jmath}$ equals $-\frac{1}{2}$.

In $\mathcal{N}=4$ SYM, there are two types of (chiral) shortening conditions for particular values of the conformal dimension Δ: BPS (B) which may occur when at least one of the spins is zero, and semi-short (C) ones. The corresponding multiplets are constructed similar to the long ones (A.1), with the sum running only over a restricted number of supersymmetries. Specifically, the critical values of the conformal dimensions and the restrictions on the sums in (A.1) are given by

$$\begin{array}{llll} \mathrm{B}_L: & \mathcal{V}^{\Delta,B}_{[0^\dagger,\bar{\jmath}][q_1,p,q_2]} & \Delta = p+\frac{3}{2}q_1+\frac{1}{2}q_2 & \epsilon_{1\pm}=0 \\ \mathrm{B}_R: & \mathcal{V}^{\Delta,B}_{[j,0^\dagger][q_1,p,q_2]} & \Delta = p+\frac{1}{2}q_1+\frac{3}{2}q_2 & \bar{\epsilon}_{4\pm}=0 \\ \mathrm{C}_L: & \mathcal{V}^{\Delta,B}_{[j^*,\bar{\jmath}][q_1,p,q_2]} & \Delta = 2+2j+p+\frac{3}{2}q_1+\frac{1}{2}q_2 & \epsilon_{1-}=0 \\ \mathrm{C}_R: & \mathcal{V}^{\Delta,B}_{[j,\bar{\jmath}^*][q_1,p,q_2]} & \Delta = 2+2\bar{\jmath}+p+\frac{1}{2}q_1+\frac{3}{2}q_2 & \bar{\epsilon}_{4-}=0 \end{array} \tag{A.4}$$

for the different types of multiplets. They represent the basic $\frac{1}{8}$-BPS and $\frac{1}{16}$ semishortenings in $\mathcal{N}=4$ SCA and are indicated as in with a "$\dagger$" and a "$*$" respectively.

If the conformal dimension Δ of the HWS of a long multiplet satisfies one of the conditions (A.4), the multiplet splits according to

$$\begin{array}{ll} \mathrm{L}: & \mathcal{V}^{\Delta,B}_{[j,\bar{\jmath}][q_1,p,q_2]} = \mathcal{V}^{\Delta,B}_{[j^*,\bar{\jmath}][q_1,p,q_2]} + \mathcal{V}^{\Delta+\frac{1}{2},B-\frac{1}{2}}_{[j-\frac{1}{2}^*,\bar{\jmath}][q_1+1,p,q_2]}\,, \\ \mathrm{R}: & \mathcal{V}^{\Delta,B}_{[j,\bar{\jmath}][q_1,p,q_2]} = \mathcal{V}^{\Delta,B}_{[j,\bar{\jmath}^*][q_1,p,q_2]} + \mathcal{V}^{\Delta+\frac{1}{2},B+\frac{1}{2}}_{[j,\bar{\jmath}-\frac{1}{2}^*][q_1,p,q_2+1]}\,, \end{array} \tag{A.5}$$

where by '$*$' we denote the 1/16 semishortening. Consequently, we denote by $\mathcal{V}^{\Delta,B}_{[j^*,\bar{\jmath}^*][q_1,p,q_2]}$ the 1/8 semi-short multiplets appearing in the decomposition

$$\begin{aligned}\mathcal{V}^{\Delta,B}_{[j,\bar{\jmath}][q_1,p,q_2]} &= \mathcal{V}^{\Delta,B}_{[j^*,\bar{\jmath}^*][q_1,p,q_2]} + \mathcal{V}^{\Delta+\frac{1}{2},B-\frac{1}{2}}_{[j-\frac{1}{2}^*,\bar{\jmath}^*][q_1+1,p,q_2]} + \mathcal{V}^{\Delta+\frac{1}{2},B+\frac{1}{2}}_{[j^*,\bar{\jmath}-\frac{1}{2}^*][q_1,p,q_2+1]} \\ &\quad + \mathcal{V}^{\Delta+1,B}_{[j-\frac{1}{2}^*,\bar{\jmath}-\frac{1}{2}^*][q_1+1,p,q_2+1]}\,, \end{aligned} \tag{A.6}$$

if left and right shortening conditions in (A.4) are simultaneously satisfied. The semishort multiplets appearing in this decomposition are constructed explicitly according to (A.1), (A.4).

Formulae (A.5) include the special cases $\mathcal{V}^{\Delta,B}_{[j^*,\bar{\jmath}][0,p,q_2]}$, $\mathcal{V}^{\Delta,B}_{[j^*,\bar{\jmath}][0,0,q_2]}$, and $\mathcal{V}^{\Delta,B}_{[j^*,\bar{\jmath}][0,0,0]}$, corresponding to (chiral) 1/8, 3/16, and 1/4 semi-shortening, respectively; likewise for $\mathcal{V}^{\Delta,B}_{[j,\bar{\jmath}^*][q_1,p,0]}$, $\mathcal{V}^{\Delta,B}_{[j,\bar{\jmath}^*][q_1,0,0]}$, and $\mathcal{V}^{\Delta,B}_{[j,\bar{\jmath}^*][0,0,0]}$. For $j = 0$ and $\bar{\jmath} = 0$, respectively, the decompositions (A.5) yield negative spin labels. They are to be interpreted as BPS multiplets, denoted by '†', as follows

$$\mathcal{V}^{\Delta,B}_{[-\frac{1}{2}^*,\bar{\jmath}][q_1,p,q_2]} \equiv \mathcal{V}^{\Delta+\frac{1}{2},B+\frac{1}{2}}_{[0^\dagger,\bar{\jmath}][q_1+1,p,q_2]} , \qquad \mathcal{V}^{\Delta,B}_{[j,-\frac{1}{2}^*][q_1,p,q_2]} \equiv \mathcal{V}^{\Delta+\frac{1}{2},B-\frac{1}{2}}_{[j,0^\dagger][q_1,p,q_2+1]} , \tag{A.7}$$

where one verifies that the BPS highest weight states satisfy the BPS shortening conditions of (A.4). In addition, there is the series $\mathcal{V}^{p,0}_{[0^\dagger,0^\dagger][0^\dagger,p,0^\dagger]}$ of $\frac{1}{2}$-BPS multiplets.

For convenience (but not quite accurately) we can also define

$$\mathcal{V}^{p,0}_{[-1^*,-1^*][0,p,0]} := \mathcal{V}^{p+2,0}_{[0^\dagger,0^\dagger][0^\dagger,p+2,0^\dagger]}. \tag{A.8}$$

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